Exponential concentration of cover times

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Outline

- Part I: Preliminaries
  - Effective resistance and Gaussian free fields
  - Ray-Knight theorems
- Part II: Application to cover times
- Part III: Stochastic domination in the generalized 2nd Ray-Knight theorem

Part I: Preliminaries

Our setting

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- Define
  - **cover time**

$$\tau_{\text{cov}} = \text{the first time all vertices are visited at least once}$$

  - **hitting time**

$$\tau_{\text{hit}}(x, y) = \text{the first time walk started at } x \text{ visits } y$$

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- We can compute $R_{\text{eff}}(x, y)$ by solving for a function $f : V \to \mathbb{R}$ such that

$$\Delta f(z) = \begin{cases} 1 & \text{if } z = x \\ -1 & \text{if } z = y \\ 0 & \text{otherwise} \end{cases}$$

Then $R_{\text{eff}}(x, y) = f(y) - f(x)$.

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- Commute time identity:

$$\frac{\mathbf{E}\tau_{\text{hit}}(x, y) + \mathbf{E}\tau_{\text{hit}}(y, x)}{2} = |E| \cdot R_{\text{eff}}(x, y).$$

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$$[\text{probability of } f] \quad \propto \quad \exp\left(-\frac{1}{2}\sum_{(x,y)\in E}(f_x - f_y)^2\right)$$

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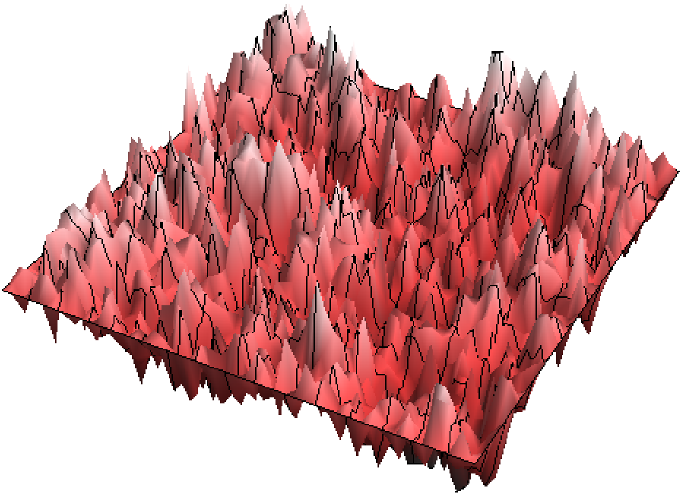
$$[\text{probability of } f] \quad \propto \quad \exp\left(-\frac{1}{2} \sum_{(x,y) \in E} (f_x - f_y)^2 \right)$$

- equivalently,

$$\mathbf{E}\,(\eta_x - \eta_y)^2 = R_{\text{eff}}(x, y) \qquad (\text{note: } \mathbf{E}\eta_x^2 = R_{\text{eff}}(x, v_0))$$

Gaussian free field: example

Below is a realization of the GFF on a discrete 2D lattice:



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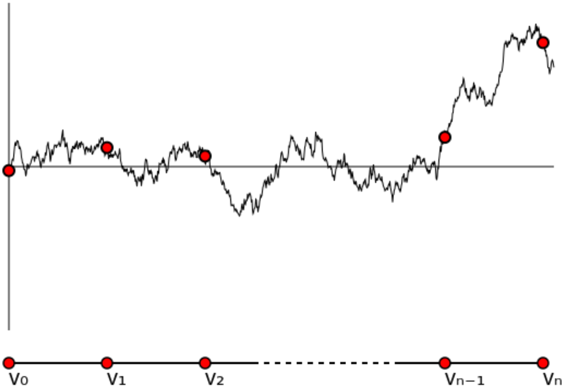

Let $\{B_t\}_{t \geq 0}$ be a Brownian motion. GFF of a path is

$$\eta = (0 = B_0, B_1, \ldots, B_n).$$

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- For $x \in V$ and $s \in \mathbb{R}^+$, define **local time**

$$\begin{aligned}\mathcal{L}_s(x) &= \frac{1}{\deg(x)} \int_0^s \mathbf{1}\left(X_{s'} = x\right) ds' \\ &= \frac{1}{\deg(x)} \left(\text{time spent by r.w. at } x \text{ up to time } s\right).\end{aligned}$$

Return times

- For any $t > 0$, define

$$\begin{aligned}\tau^+(t) &= \inf\{s \geq 0 : \mathcal{L}_s(v_0) \geq t\} \\ &= \text{first time that } v_0 \text{ accumulates local time } t.\end{aligned}$$

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- Remark: $\tau^+\left(\frac{1}{\deg(v_0)}\right)$ is like the return time of a discrete time random walk.
- We have

$$\mathbf{E}\tau^+(t) = 2|E| \cdot t.$$

(Analogous to expected return time being equal to inverse stationary probability.)

Generalized 2nd Ray-Knight theorem

Theorem (Generalized Second Ray-Knight Theorem)

Let X be a continuous time random walk, and let η and η' be GFFs with X and η independent. Then, for any $t > 0$,

$$\left\{ \mathcal{L}_{\tau^+(t)}(x) + \frac{1}{2}\eta_x^2 \right\}_{x \in V} \overset{law}{=} \left\{ \frac{1}{2}\left(\eta'_x + \sqrt{2t}\right)^2 \right\}_{x \in V}.$$

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Above theorem due to Eisenbaum-Kaspi-Marcus-Rosen-Shi.

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Similar/related theorems by Ray, Knight, Dynkin, Le Jan, Sznitman, and others.

Part II: Application to cover times

Gaussian concentration inequality

Theorem (Borell and Sudakov-Tsirelson)

Let $\eta = \{\eta_i\}_{i \in I}$ be any centered multivariate Gaussian with $\mathbf{E}\eta_i^2 \le \sigma^2$ for each i. Let

$$X = \sup_{i \in I} \eta_i.$$

Then,

$$\mathbf{P}\left(|X - \mathbf{E}X| > s \cdot \sigma\right) \le 2\left(1 - \Phi(s)\right),$$

where Φ is the Gaussian CDF.

In other words, the maximum (or minimum) of a Gaussian process is at least as concentrated as a Gaussian.

Fluctuations of the GFF

- Define

$$R = \max_{x,y \in V} R_{\text{eff}}(x, y) \geq \max_{x \in V} \mathbf{E}\eta_x^2$$

$$M = \mathbf{E} \max_{v \in V} \eta_v = -\mathbf{E} \min_{v \in V} \eta_v.$$

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- By Gaussian concentration, $\max_{v \in V} \eta_v$ has mean M and fluctuations of order $\sqrt{R}$.
- In many cases, $\sqrt{R} \ll M$.
  - e.g. complete graph, discrete torus, regular trees
  - **doesn't** hold for case of a path

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$$\left\{\mathcal{L}_{\tau^+(t)}(x) + \frac{1}{2}\eta_x^2\right\}_{x\in V} \overset{law}{=} \left\{\frac{1}{2}\left(\eta'_x + \sqrt{2t}\right)^2\right\}_{x\in V}$$

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Main observation (Ding-Lee-Peres):

$$\tau^+(t) < \tau_{\mathrm{cov}} \iff \text{one of the } \mathcal{L}_{\tau^+(t)}(x) \text{ is } 0$$

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Theorem (Ding-Lee-Peres)

$$\mathbf{E}\tau_{cov} \asymp |E| \cdot \left(-\mathbf{E} \min_{x \in V} \eta'_x \right)^2 = |E| \cdot M^2.$$

Statement of the concentration bound

Theorem (Z., following conjecture of Ding)

There are universal constants c and C such that

$$\mathbf{P}\left(\left|\tau_{cov} - |E|M^2\right| \geq |E|(\sqrt{\lambda R} \cdot M + \lambda R)\right) \leq Ce^{-c\lambda}$$

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Recall:

$$\max_{x,y\in V} \mathbf{E}\tau_{\text{hit}}(x,y) \asymp |E|\cdot R \qquad \text{and} \qquad \mathbf{E}\tau_{\text{cov}} \asymp |E|\cdot M^2.$$

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(Ding proved for trees and bounded degree graphs.)

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Suppose $\mathbf{P}\left(\mathcal{L}_{\tau^+(t)}(x) = 0 \text{ for some } x\right)$ is large.

Upper bound (following Ding-Lee-Peres)

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is large, which means $\sqrt{2t}$ can't be much more than M.

Lower bound (following Ding)

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Important missing step: how to make “$\implies$” rigorous.

Concentration of cover times: recap

- The transition point of whether

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occurs around $\sqrt{2t} \approx M \implies t \approx \frac{1}{2}M^2$.

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- $\tau^+(t)$ is concentrated around its expectation $2|E| \cdot t$ as long as $R \ll t$, so

$$\tau_{\text{cov}} \approx \tau^+\left(\frac{1}{2}M^2\right) \approx |E| \cdot M^2.$$

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- But still need “important missing step”.

Part III: Stochastic domination in the generalized 2nd Ray-Knight theorem

Stochastic domination in the second Ray-Knight theorem

Theorem (variant of theorem of Lupu, conjectured by Ding)

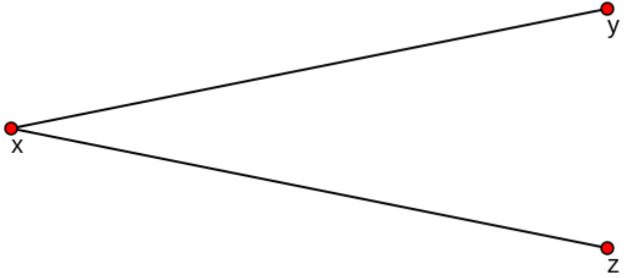
We have

$$\left\{\sqrt{\mathcal{L}_{\tau^+(t)}(x)} : x \in V\right\} \preceq \frac{1}{\sqrt{2}}\left\{\max\left(\eta'_x + \sqrt{2t}, 0\right) : x \in V\right\},$$

where $\preceq$ denotes stochastic domination.

A graph refinement

Random walk step can be simulated by random walk on refined graph:



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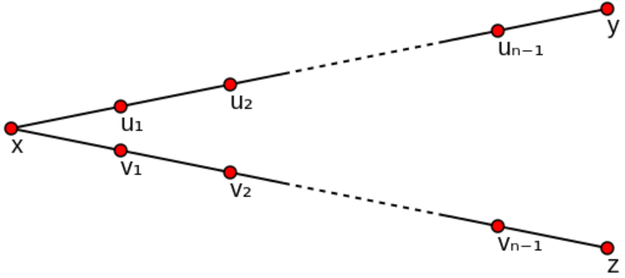

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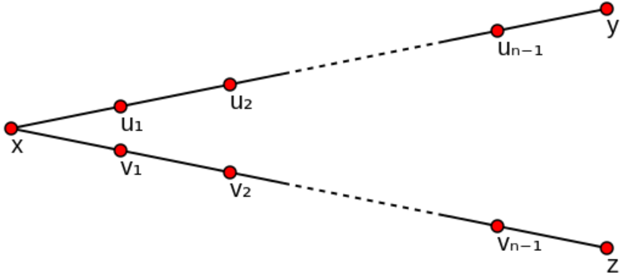

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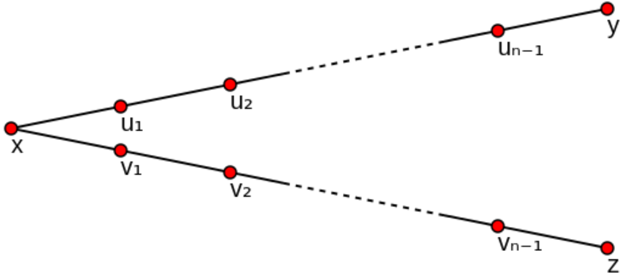

Refined walk visits x a $\mathbf{Geom}(n)$ number of times before going to y or z with equal probability $\implies$ time spent at x is still exponential

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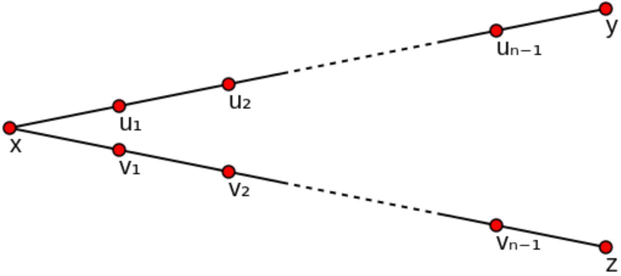
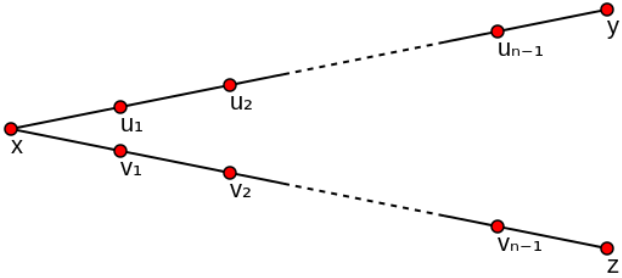


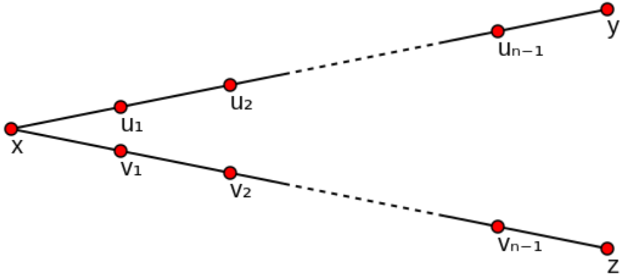
The GFFs are also related in a natural way: effective resistances (= GFF covariances) are multiplied by n.

Metric graphs



The limiting object as $n \to \infty$ is known as a **metric graph**.

Metric graphs



The limiting object as $n \to \infty$ is known as a **metric graph**. In the limit:

- random walk is a "Brownian motion on edges".
- GFF has same law as original graph (up to scaling), with Brownian bridges on edges

Proof of stochastic domination

Artificially construct a coupling of random walk X and GFFs η and η' on the metric graph so that

$$\left\{ \mathcal{L}_{\tau^+(t)}(x) + \frac{1}{2}\eta_x^2 \right\}_{x \in V} = \left\{ \frac{1}{2} \left(\eta'_x + \sqrt{2t} \right)^2 \right\}_{x \in V}.$$

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Claim: U is (a.s.) connected.
- $\eta'_x + \sqrt{2t} = 0$ forces $\mathcal{L}_{\tau^+(t)}(x) = 0$
- $\eta'_x + \sqrt{2t}$ can't change signs on U and is positive at $x = v_0$

Some open questions

Theorem (generalized Ray-Knight)

$$\left\{ \mathcal{L}_{\tau^+(t)}(x) + \frac{1}{2}\eta_x^2 \right\}_{x\in V} \overset{law}{=} \left\{ \frac{1}{2}\left(\eta'_x + \sqrt{2t}\right)^2 \right\}_{x\in V}$$

- Only known proofs are by moment calculations. Can we give an explicit coupling?
- Can be understood relatively well when graph is a path or tree. What about a cycle?

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