Chaining introduction

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Given a collection of random variables $X_1, X_2, \ldots,$, we would like to say that $\max_i X_i$ is small with high probability. (Happens all over computer science, e.g. Chernoff+Union bound)

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Today's topic: Beating the Union Bound

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Disclaimer: This is a tutorial talk, about ideas which aren't mine.

Chaining applications in computer science

- Analyzing structured RIP matrices for compressed sensing (Candès, Tao'06), (Rudelson, Vershynin'06), (Cheragchi, Guruswami, Velingker'13), (N., Price, Wootters'14), (Bourgain'14), (Haviv, Regev'15)
- Fast Johnson-Lindenstrauss (JL) transforms (Ailon, Liberty'11), (Krahmer, Ward'11), (Bourgain, Dirksen, N.'15), (Oymak, Recht, Soltanolkotabi'15)
- Instance-wise JL bounds (Gordon'88), (Klartag, Mendelson'05), (Mendelson, Pajor, Tomczak-Jaegermann'07), (Dirksen'14)
- Approximate nearest neighbor (Indyk, Naor'07)
- Deterministic alg. to estimate graph cover time (Ding, Lee, Peres'11), (Zhai'14)
- List-decodability of random codes (Wootters'13), (Rudra, Wootters'14)
- Streaming heavy hitters (Braverman et al.'16), (Braverman et al.'17?)
- Dictionary learning (Luh, Vu'15), (Błasiok, Nelson'16)
- ...

Case study for this talk:

gaussian processes

Gaussian process setup

- $T \subset B_{\ell_2^n}$

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- Define *gaussian mean width* $\mathfrak{g}(T) = \mathbb{E}_g \sup_{x \in T} Z_x$

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- How can we bound $\mathfrak{g}(T)$?

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- **This talk:** four progressively tighter ways to bound $\mathfrak{g}(T)$, then discussion of lower bounds.

Gaussian mean width bound 1: union bound

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$$\begin{aligned}\mathbb{E}\sup_{x\in T} Z_x &\le \int_0^\infty \mathbb{P}(\sup_{x\in T} Z_x > u)du \\ &= \int_0^{u_*} \underbrace{\mathbb{P}(\sup_{x\in T} Z_x > u)}_{\le 1} du + \int_{u_*}^\infty \underbrace{\mathbb{P}(\sup_{x\in T} Z_x > u)}_{\le |T|\cdot e^{-u^2/2} \text{ (union bound)}} du\end{aligned}$$

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&= \int_0^{u_*} \underbrace{\mathbb{P}(\sup_{x \in T} Z_x > u)}_{\le 1} du + \int_{u_*}^\infty \underbrace{\mathbb{P}(\sup_{x \in T} Z_x > u)}_{\le |T| \cdot e^{-u^2/2} \text{ (union bound)}} du \\
&\le u_* + |T| \cdot e^{-u_*^2/2} \\
&\lesssim \sqrt{\log |T|} \text{ (set } u_* = \sqrt{2 \log |T|})
\end{aligned}
$$

Gaussian mean width bound 2: ε-net

- $\mathfrak{g}(T) = \mathbb{E} \sup_{x \in T} \langle g, x \rangle$
- Let S_ε be ε-net of (T, ℓ_2) (every $x \in T$ is ε-close to some $x' \in S_\varepsilon$ in ℓ_2)

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- $\langle g, x \rangle = \langle g, x' \rangle + \langle g, x - x' \rangle$ $(x' = \operatorname{argmin}_{y \in S_\varepsilon} \|x - y\|_2)$

  $\mathfrak{g}(T) \leq \mathfrak{g}(S_\varepsilon) + \mathbb{E}_g \sup_{x \in T} \underbrace{\langle g, x - x' \rangle}_{\leq \varepsilon \cdot \|g\|_2}$

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  $\mathfrak{g}(T) \le \mathfrak{g}(S_\varepsilon) + \mathbb{E}_g \sup_{x\in T}\underbrace{\langle g, x - x'\rangle}_{\le \varepsilon\cdot\|g\|_2}$
- $\lesssim \sqrt{\log|S_\varepsilon|} + \varepsilon(\mathbb{E}_g\|g\|_2^2)^{1/2}$
- $\lesssim \log^{1/2}\underbrace{\mathcal{N}(T, \ell_2, \varepsilon)}_{\text{smallest } \varepsilon-\text{net size}} + \varepsilon\sqrt{n}$

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  $\mathfrak{g}(T) \le \mathfrak{g}(S_\varepsilon) + \mathbb{E}_g \sup_{x \in T} \underbrace{\langle g, x - x' \rangle}_{\le \varepsilon \cdot \|g\|_2}$
- $\lesssim \sqrt{\log |S_\varepsilon|} + \varepsilon (\mathbb{E}_g \|g\|_2^2)^{1/2}$
- $\lesssim \log^{1/2} \underbrace{\mathcal{N}(T, \ell_2, \varepsilon)}_{\text{smallest } \varepsilon-\text{net size}} + \varepsilon \sqrt{n}$
- Choose ε to optimize bound; can never be worse than last slide (which amounts to choosing $\varepsilon = 0$)

Gaussian mean width bound 3: ε-net sequence

- S_r is a $({}^1\!/_{2^r})$-net of T, $r \geq 0$
  $\pi_r x$ is closest point in S_r to $x \in T$, $\Delta_r x = \pi_r x - \pi_{r-1} x$

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- S_r is a $(1/2^r)$-net of T, $r \geq 0$
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- wlog $|T| < \infty$ (else apply this slide to ε-net of T for ε small)
- $x = \pi_0 x + (\pi_1 x - \pi_0 x) + (\pi_2 x - \pi_1 x) + \ldots = \pi_0 x + \sum_{r=1}^{\infty} \Delta_r x$

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- $$|\{\Delta_r x : x \in T\}| \leq \mathcal{N}(T, \ell_2, 1/2^r) \cdot \mathcal{N}(T, \ell_2, 1/2^{r-1}) \leq (\mathcal{N}(T, \ell_2, 1/2^r))^2$$
  $\sigma_r = \sup_{x \in T} \|\Delta_r x\|_2 = \sup \|\pi_r x - x + x - \pi_{r-1} x\| \leq \frac{1}{2^r} + \frac{1}{2^{r-1}}$

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- $\mathfrak{g}(T) \lesssim \sum_{r=1}^{\infty} (1/2^r) \cdot \log^{1/2} \mathcal{N}(T, \ell_2, 1/2^r)$
  $\lesssim \int_0^{\infty} \log^{1/2} \mathcal{N}(T, \ell_2, u) du$ (Dudley's theorem)

Gaussian mean width bound 4: generic chaining

- Again, wlog $|T| < \infty$. Define $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{k_*} = T$
  $|T_0| = 1, |T_k| \leq 2^{2^k}$ (call such a sequence "admissible")

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- Exercise: show Dudley's theorem is equivalent to
  $\mathfrak{g}(T) \lesssim \inf_{\{T_k\} \text{ admissible}} \sum_{k=1}^{\infty} 2^{k/2} \cdot \sup_{x \in T} d_{\ell_2}(x, T_k)$
  (should pick T_k to be the best $\varepsilon = \varepsilon(k)$ net of size 2^{2^k})

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- Fernique'76*: can pull the $\sup_x$ *outside* the sum
- $\mathfrak{g}(T) \lesssim \inf_{\{T_k\}} \sup_{x \in T} \sum_{k=1}^{\infty} 2^{k/2} \cdot d_{\ell_2}(x, T_k) \stackrel{\text{def}}{=} \gamma_2(T, \ell_2)$

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  $*$ equivalent upper bound proven by Fernique (who minimized some integral over all measures over T), but reformulated in terms of admissible sequences by Talgarand

Gaussian mean width bound 4: generic chaining

Proof of Fernique's bound

$$\mathfrak{g}(T) \leq \underbrace{\mathbb{E}_g \sup_{x \in T} \langle g, \pi_0 x \rangle}_{0} + \mathbb{E}_g \sup_{x \in T} \sum_{k=1}^{\infty} \underbrace{\langle g, \Delta_k x \rangle}_{Y_k(x)} \text{ (from before)}$$

- $\forall t$, $\mathbb{P}(Y_k(x) > t 2^{k/2} \|\Delta_k x\|_2) \leq e^{-t^2 2^k/2}$ (gaussian decay)

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$$\mathop{\mathbb{E}}_{g} \sup_{x\in T} \sum_k Y_k(x) \le \int_0^{\infty} \mathbb{P}(\sup_{x\in T} \sum_k Y_k(x) > u) du$$

Gaussian mean width bound 4: generic chaining

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Gaussian mean width bound 4: generic chaining

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$$= \gamma_2(T, \ell_2) \cdot \int_0^\infty \mathbb{P}(\sup_{x \in T} \sum_k Y_k(x) > t \sup_{x \in T} \sum_k 2^{k/2} \|\Delta_k x\|_2) dt$$

$$\text{(change of variables: } u = t \sup_{x \in T} \sum_k 2^{k/2} \|\Delta_k x\|_2 \simeq t\gamma_2(T, \ell_2))$$

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(change of variables: $u = t \sup_{x\in T} \sum_k 2^{k/2} \|\Delta_k x\|_2 \simeq t\gamma_2(T,\ell_2)$)

$$\le \gamma_2(T,\ell_2) \cdot [2 + \int_2^\infty \left(\sum_{k=1}^\infty (2^{2^k})^2 e^{-t^2 2^k/2} \right) dt]$$

$$= \gamma_2(T,\ell_2) \cdot [2 + \sum_{k=1}^\infty \left(\int_2^\infty (2^{2^k})^2 e^{-t^2 2^k/2} dt \right)] \simeq \gamma_2(T,\ell_2)$$

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$$= \gamma_2(T,\ell_2)\cdot [2 + \sum_{k=1}^\infty \left(\int_2^\infty (2^{2^k})^2 e^{-t^2 2^k/2} dt\right)] \simeq \gamma_2(T,\ell_2)$$

- Conclusion: $\mathfrak{g}(T) \lesssim \gamma_2(T,\ell_2)$

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$$= \gamma_2(T,\ell_2) \cdot \int_0^\infty \mathbb{P}(\sup_{x\in T} \sum_k Y_k(x) > t \sup_{x\in T} \sum_k 2^{k/2} \|\Delta_k x\|_2) dt$$

(change of variables: $u = t \sup_{x\in T} \sum_k 2^{k/2}\|\Delta_k x\|_2 \simeq t\gamma_2(T,\ell_2)$)

$$\le \gamma_2(T,\ell_2) \cdot [2 + \int_2^\infty \left(\sum_{k=1}^\infty (2^{2^k})^2 e^{-t^2 2^k/2} \right) dt]$$

$$= \gamma_2(T,\ell_2) \cdot [2 + \sum_{k=1}^\infty \left(\int_2^\infty (2^{2^k})^2 e^{-t^2 2^k/2} dt \right)] \simeq \gamma_2(T,\ell_2)$$

- Conclusion: $\mathfrak{g}(T) \lesssim \gamma_2(T,\ell_2)$
- Talagrand: $\mathfrak{g}(T) \simeq \gamma_2(T,\ell_2)$ (will show soon) ("Majorizing measures theorem")

Are these bounds really different?

- $\gamma_2(T, \ell_2)$**:** $\inf_{\{T_k\}} \sup_{x \in T} \sum_{k=1}^{\infty} 2^{k/2} \cdot d_{\ell_2}(x, T_k)$
- **Dudley:** $\inf_{\{T_k\}} \sum_{k=1}^{\infty} 2^{k/2} \cdot \sup_{x \in T} d_{\ell_2}(x, T_k)$
  $\simeq \int_0^{\infty} \log^{1/2} \mathcal{N}(T, \ell_2, u) du$

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- Simple vanilla ε-net argument gives $\mathfrak{g}(B_{\ell_1^n}) \lesssim \mathrm{poly}(n)$.

High probability

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- Moment method: bound $\mathbb{E}_g \sup_{x \in T} Z_x^p$ for large p and do Markov

  Can bound moments using chaining too; see talk by Dirksen

  (also see theorem of Borell, Ibragimov, Sudakov, Tsirelson [Theorem 5.8 of "Concentration Inequalities" by Boucheron, Lugosi, Massart])

Lower Bounds

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Lemma (Slepian)

Let $z_1, \ldots, z_n$ and $z'_1, \ldots, z'_n \in \mathbb{R}^N$ be such that for all t, t'

$$\|z_t - z_{t'}\|_2 \geq \|z'_t - z'_{t'}\|_2$$

Let $X_t = \langle g, z_t \rangle$, $Y_t = \langle g, z'_t \rangle$. Then

$$\forall u_1, \ldots, u_n \in \mathbb{R}, \; \mathbb{P}\Big(\bigwedge_{t=1}^{n} (X_t \leq u_t)\Big) \leq \mathbb{P}\Big(\bigwedge_{t=1}^{n} (Y_t \leq u_t)\Big)$$

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Won't prove today. Can find three different proofs in:

- Ledoux-Talagrand book, "Probability in Banach Spaces" (Corollary 3.14 — loses a factor of 2 in the conclusion)
- Ramon van Handel book, "Probability in High Dimensions" (Thm 6.8)
- Mikahil Gromov's paper, "Monotonicity of the volume of intersection of balls"

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Suppose $T \subset \mathbb{R}^N$ contains points $z_1, \ldots, z_n$ such that $\|z_i - z_j\|_2 \geq \alpha$ for all $i \neq j$. Then $\mathfrak{g}(T) \gtrsim \alpha\sqrt{\log n}$.

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Proof.

Use Slepian with $z'_1, \ldots, z'_n = (\alpha/\sqrt{2})e_1, \ldots, (\alpha/\sqrt{2})e_n$. Then $\mathfrak{g}(T) \geq \frac{\alpha}{\sqrt{2}} \cdot \mathbb{E} \max\{g_1, \ldots, g_n\} \gtrsim \alpha\sqrt{\log n}$. □

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- **Sudakov:** $\mathfrak{g}(T) \gtrsim \sup_{r \geq 1} \frac{1}{2^r} \log^{1/2} \mathcal{D}(T, \ell_2, \frac{1}{2^r})$

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Thus, essentially Sudakov is the largest term in a sum and Dudley is the sum of all terms. Can show only $k = O(\log N)$ matters for dimension N (the rest of the sum is $O(1)$), so $O(\log N)$ gap (similar gap between Sudakov and $\gamma_2(T, \ell_2)$).

Strengthening Sudakov

Theorem (Sudakov++)

Suppose $T \subset \mathbb{R}^N$ *contains points* $x_1, \ldots, x_n$ *such that* $\|x_i - x_j\|_2 \geq \alpha$ *for all* $i \neq j$. *Then for some constants* $r \geq 4, \kappa > 0$
$\mathfrak{g}(T) \geq \kappa\alpha\sqrt{\log n} + \min_{i=1,\ldots,n} \mathfrak{g}(B_{\ell_2}(x_i, \alpha/r) \cap T)$.

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$$Q_i = \sup_{x \in B_{\ell_2}(x_i, \alpha/r) \cap T} Z_x - Z_{x_i}$$

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&\geq c\alpha\sqrt{\log n} + \min_i \mathfrak{g}(B(x_i, \alpha/r) \cap T) - c' \frac{\alpha}{r} \sqrt{\log n}
\end{aligned}$$

Pick $r > 2c'/c$. (Last line: gaussian concentration of Lipschitz functions.) □

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Function $f : \mathbb{R}^n \to \mathbb{R}$ is *L-Lipschitz* if for all $x, y \in \mathbb{R}^n$,

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If f is L-Lipschitz, then for all $\lambda > 0$

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- $\|x - x_i\|_2 \le \alpha/r$, so f (α/r)-Lipschitz by Cauchy-Schwarz
- Hence $\mathbb{E} \max_{i=1,\ldots,n} |Q_i - \mathbb{E}\, Q_i| \le c'(\alpha/r)\sqrt{\log n}$.

Using Sudakov++ to prove majorizing measures

(based on notes by James Lee, itself based on book by Talagrand)

$$\gamma_2(T, \ell_2) \leq \inf_{\substack{\{T_k\} \\ admissible}} \sup_{x \in T} \sum_{k=0}^{\infty} 2^{k/2} \cdot diam(\mathcal{P}_k(x))$$

where *diam* is ℓ_2 diameter, and $\mathcal{P}_k(x)$ is the set of all points in T whose nearest neighbor in T_k is the same as x's nearest neighbor.

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- cost of tree $\mathcal{T}$ is $cost(\mathcal{T}) = \sup_{x \in T} \sum_{k=0}^{\infty} 2^{k/2} \cdot diam(\mathcal{P}_k(x))$ will show $\forall T\ \exists \mathcal{T},\ \mathfrak{g}(T) \gtrsim cost(\mathcal{T})$

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- Note if P is root-to-leaf path for $x \in T$, $\sum_k 2^{k/2}\, diam(\mathcal{P}_k(x)) \lesssim value(P)$

Lower bound from greedy tree

- Label node A with *value* $\mathfrak{g}(A)$; each edge to child with value $\kappa \frac{rad(A)}{r} \cdot 2^{k/2}$.
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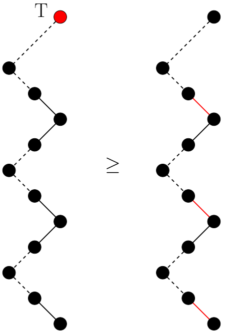
- Thus for any P, its value is $\approx$ same only considering last right turns

Desired inequality

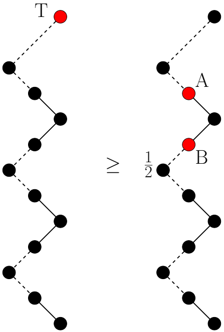


- Inequality is over values in red, so want $\mathfrak{g}(T) \geq \mathit{value}(P)$ (only counting last right edges)

*Figures and presentation adapted from notes of James Lee

Proof of tree lower bound

Lemma 1:



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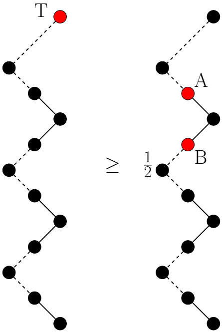
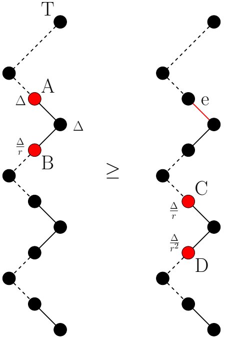


- $B \subseteq A \subseteq T$, so $\mathfrak{g}(T) \geq \mathfrak{g}(A) \geq \mathfrak{g}(B)$
  $\Rightarrow \mathfrak{g}(T) \geq \frac{1}{2}(\mathfrak{g}(A) + \mathfrak{g}(B))$

Proof of tree lower bound

Lemma 2:



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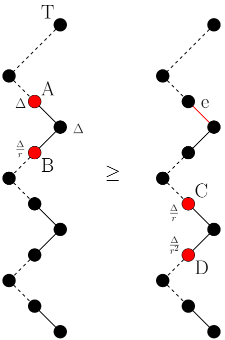
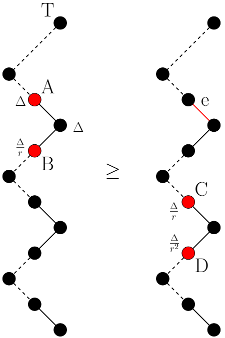


- $\mathfrak{g}(T) \geq \frac{1}{2}$(sum of last right turns) follows by applying Lemma 1 once then iterating Lemma 2 down the path.

Proof of tree lower bound

Lemma 2:



- $\mathfrak{g}(B) \geq \mathfrak{g}(C)$ since $C \subseteq B$. Now must show $value(A) \geq value(e) + value(D) = \kappa \frac{\Delta}{r} 2^{k/2} + \mathfrak{g}(D)$.

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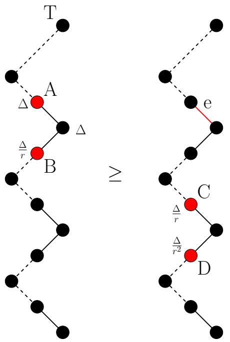
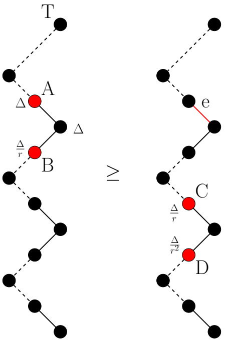


- $\mathfrak{g}(B) \geq \mathfrak{g}(C)$ since $C \subseteq B$. Now must show $value(A) \geq value(e) + value(D) = \kappa \frac{\Delta}{r} 2^{k/2} + \mathfrak{g}(D)$.
- A refined into $m = 2^{2^k}$ pieces. Since carved out balls of radius Δ/r, ball centers x_i are $\frac{\Delta}{r}$-far apart and can apply Sudakov++.
  $\mathfrak{g}(A) \geq \kappa \frac{\Delta}{r} \sqrt{\log m} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A) =$
  $\kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A)$

Proof of tree lower bound

Lemma 2:



$$\mathfrak{g}(A) \quad \geq \quad \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A)$$

Proof of tree lower bound

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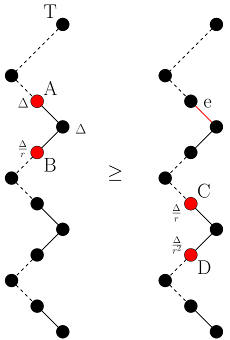
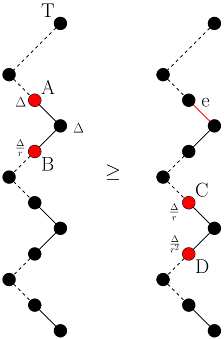


$$\mathfrak{g}(A) \geq \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A)$$

$$\geq \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap D_i) \ (D_i \subseteq A)$$

Proof of tree lower bound

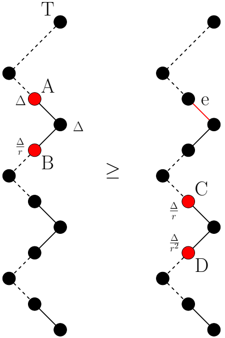
Lemma 2:



$$\begin{aligned} \mathfrak{g}(A) &\geq \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A) \\ &\geq \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap D_i) \ (D_i \subseteq A) \\ &= \kappa \frac{\Delta}{r} 2^{k/2} + \mathfrak{g}(B(x_m, \Delta/r^2) \cap D_m) \text{ (by greedy construction)} \end{aligned}$$

Proof of tree lower bound

Lemma 2:



$$
\begin{aligned}
\mathfrak{g}(A) &\geq \kappa \frac{\Delta}{r} 2^{k/2} + \min_i \mathfrak{g}(B(x_i, \Delta/r^2) \cap A) \\
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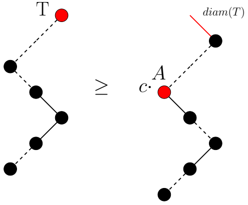
&= \kappa \frac{\Delta}{r} 2^{k/2} + \mathfrak{g}(B(x_m, \Delta/r^2) \cap D_m) \text{ (by greedy construction)} \\
&= \kappa \frac{\Delta}{r} 2^{k/2} + \mathfrak{g}(D) \; (D \subseteq D_m \text{ of radius } \leq \frac{\Delta}{r^2}, \text{ and } x_m \text{ is maximizer})
\end{aligned}
$$

Issues at the boundary

What if the first turn in P is a left turn?



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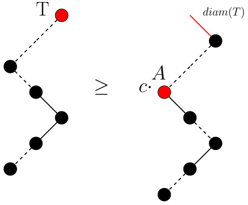

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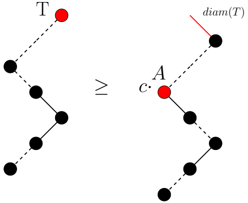

- $\mathfrak{g}(T) = \frac{1}{2}(\mathfrak{g}(T) + \mathfrak{g}(T))$
  $\geq \frac{1}{2}(\mathfrak{g}(T) + \mathfrak{g}(A))$ $(A \subseteq T)$

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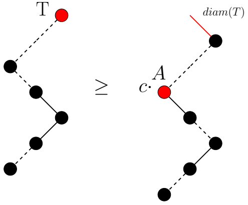
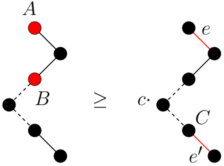


- $\mathfrak{g}(T) = \frac{1}{2}(\mathfrak{g}(T) + \mathfrak{g}(T))$
  $\geq \frac{1}{2}(\mathfrak{g}(T) + \mathfrak{g}(A))$ $(A \subseteq T)$
  $\gtrsim \frac{1}{2}(diam(T) + \mathfrak{g}(A))$ ($\mathfrak{g}(T) \gtrsim diam(T)$ always)

Issues at the boundary

What if the last turn in P is a right turn?



Issues at the boundary

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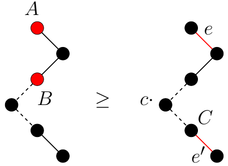

- $\mathfrak{g}(A) \gtrsim$ *value*(e) (vanilla Sudakov)

Issues at the boundary

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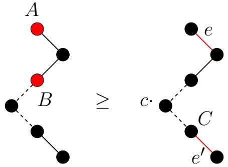

- $\mathfrak{g}(A) \gtrsim \textit{value}(e)$ (vanilla Sudakov)
- $\mathfrak{g}(B) \geq \mathfrak{g}(C)$ $(C \subseteq B)$
  then $\mathfrak{g}(C) \gtrsim \textit{value}(e')$ (vanilla Sudakov again)

The End

Some items to read

- "Notes on Gaussian processes and majorizing measures", by James Lee (see his website and tcsmath.org blog)
- "Probability in High Dimensions", by Ramon van Handel
- "Upper and lower bounds for stochastic processes: modern methods and classical problems", by Michel Talagrand