How to construct chaining algorithms and compare Bernoulli processes

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  For any centred separable Gaussian process $\bar{G}_s$, $s \in S$ there exists mapping $\varphi : S \to \varphi(S) = T \subset \ell^2$ such that

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- Denote $b(T) = \mathbf{E} \sup_{t \in T} X_t$.

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- Clearly $d_1(s,t) = d(s,t) = \|t - s\|_2$. Let also $\Delta_n(A) = \sup_{s,t \in A} d_n(s,t)$.

Functional $\gamma_2(T)$

- Let $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ is an **admissible sequence of partitions** if the partitions are nested and $|\mathcal{A}_n| \leqslant 2^{2^n}$, $n \geqslant 1$, $|\mathcal{A}_0| = 1$. Denote $N_n = 2^{2^n}$.

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- **Theorem (Talagrand)**

There exists a universal constant K such that
$$K^{-1}\gamma_2(T) \leqslant g(T) \leqslant K\gamma_2(T).$$

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- **Theorem (WB, Latala)**

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$$K^{-1} \inf_{T_1 + T_2 \supset T} (\sup_{t \in T_1} \|t\|_1 + \gamma_2(T_2))$$
$$\leqslant b(T) \leqslant K \inf_{T_1 + T_2 \supset T} (\sup_{t \in T_1} \|t\|_1 + \gamma_2(T_2)).$$

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- An **admissible sequence of nets** is a family $\mathcal{T} = (T_n)_{n\geqslant 0}$ of finite subsets of T, such that $T_n \subset T_{n+1}$ and $|T| \leqslant N_n = 2^{2^n}$ for $n \geqslant 1$ and $|T_0| = 1$.

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- Let $\pi_n(t) \in T_n$
$$\bar{\gamma}_Z = \inf_{\mathcal{T},\pi} \sup_{t\in T} \sum_{n\geqslant 1} \|Z_{\pi_{n-1}(t)} - Z_{\pi_n(t)}\|_{2^n}.$$

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- By the choice $\pi_n(t) = t^{A_n(t)}$ and $T_n = \{t^A : A \in \mathcal{A}_n\}$ we get that $\bar{\gamma}_Z(T) \leqslant 2\gamma_Z(T)$.

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- For example $T = \{e^i : i \geqslant 1\}$- basis of ℓ^2 then

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- On the other hand $\bar{\gamma}_X(T) \sim 1$.

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- We say that functionals F_j, $j \in \mathbb{Z}$ satisfies **growth condition** for $r \geqslant 4$ if for all $t \in T$ if $t^1, t^2, \ldots, t^m \in B(t, r^{-j})$, $m \geqslant 1$ are r^{-j-1} separated i.e. $\|t^l - t^k\|_2 \geqslant r^{-j-1}$ then $A \subset B(t, r^{-j})$ and $H_i \subset (B(t^i, r^{-j-2})) \cap B(t, r^{-j})$

$$F_j(\bigcup_{i=1}^{m} H_i) \geqslant \sqrt{\log m}\, r^{-j-1} + \min_{1\leqslant i\leqslant m} F_{j+2}(H_i). \qquad (1)$$

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- In particular due to the Sudakov minoration $F_j(A) = K\mathbf{E}\sup_{t\in A} G_t$ satisfies (1).

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  Suppose that a family of functional F_j, $j \in \mathbb{Z}$ satisfies (1) then for any probability measure μ on T we have

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- Hence in particular $\gamma_2(T) \leqslant K(r)(F_{j_0}(T) + \Delta(T))$.
- Note that if $|T| = N$ then it suffices that (1) is satisfied for $j_0 \leqslant j \leqslant j_1 = j_0 + \frac{n_1}{2\log r}$, where $n_1 = \log\log N$.

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- Then $F_j(A) = \sup_{t \in A} f_j(t)$, $j_0 \leqslant j \leqslant j_1$ satisfy (1) and thus
$$\gamma_2(T) \leqslant K(r)(\sup_{t \in T} f_{j_0}(t) + \Delta(T)).$$

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- To prove that $\gamma_2(T) \leqslant K(r)(\sup_{t \in T} f_{j_0}(t) + \Delta(T))$ we have to check that f_j, $j_0 \leqslant j \leqslant j_1$ satisfy conditions (2) and (3).

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- For each t^i there exists $\pi_j(t^i) \in X$ such that $\|t^i - \pi_j(t^i)\|_2 \leqslant \frac{1}{4} r^{-j-1}$.
- We prove by induction that $f_{j+2}(t^k) \leqslant f_{j+2}(x^{l_k})$, indeed $f_{j+2}(t^1) \leqslant \sup_{s \in B(t, 2r^{-j})} f_{j+2}(s) \leqslant f_{j+2}(x^{l_1})$.

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- Therefore $f_j(t) \geqslant \sqrt{\log m} r^{-j-1} + \min\{f_{j+2}(t^i): \; i = 1, 2, \ldots, m\}$, and so (3) follows.

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- Therefore $\sup_{t\in T} f_{j_0}(t) \leqslant K\gamma_2(T)$ and hence finally $\sup_{t\in T} f_{j_0}(t) \sim \gamma_2(T)$.

General partition scheme

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- But also

$$B_{n+\kappa}(t, 2^{n+\kappa} r^{-j-2}) \subset B_n(t, \frac{2^{n+\kappa} r^{-j-2}}{(1+\varepsilon)^\kappa}) \subset B_n(t, \frac{2^{n+2} r^{-j-1}}{(1+\varepsilon)^\kappa}).$$

Growth condition

- Let $F_{n,j}$, $n \geqslant 0$, $j \in \mathbb{Z}$ be a family of positive, decreasing set functionals, i.e. if $A \subset B$, $n \geqslant 0$, $j \in \mathbb{Z}$ then

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- Functionals $F_{n,j}$, $n \geqslant 0$, $j \in \mathbb{Z}$ satisfy a **growth condition** for n_0 and $r = 2^{\kappa-2} \geqslant 4$ if for any given $n \geqslant n_0$, $j \in Z$, and $t \in T$ if there exists points $t^1, t^2, \ldots t^{N_n} \in B_n(t, 2^n r^{-j})$ that are $2^{n+1} r^{-j-1}$ in d_{n+1} separated then for any sets $H_i \subset B_{n+\kappa}(t^i, 2^{n+\kappa} r^{-j-2})$ we have

$$F_{n,j}(\bigcup_{i=1}^{N_n} H_i) \geqslant 2^n r^{-j-1} + \min_{1 \leqslant i \leqslant N_n} F_{n+2,j+2}(H_i). \tag{5}$$

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- Note that $H_i \subset B_{n+\kappa}(t^i, 2^{n+\kappa} r^{-j-2}) = B_n(t^i, \frac{2^n r^{-j-1}}{(1+\varepsilon)^\kappa})$ and hence H_i are 'small'.

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- Hence for example $\gamma_Z(T) \leqslant L(r)(F_{n_0,j_0}(T) + \Delta_{n_0}(T))$.

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- Hence for example $\gamma_Z(T) \leqslant L(r)(F_{n_0,j_0}(T) + \Delta_{n_0}(T))$.
- If $|T| = N$ then it suffices that $F_{n,j}$ satisfies (4), (5) only for $n_0 \leqslant n \leqslant n_1 = \lceil \log\log N \rceil$ and $j_0 \leqslant j \leqslant j_1 = j_0 + \left\lceil \frac{n_1 - n_0}{\log r} \right\rceil$.

Canonical processes

- Canonical processes based on e.g. log-concave independent symmetric r.v.'s satisfy Sudakov minoration i.e. if $2d_n(t^l, t^k) \geqslant d_{n+1}(t^l, t^k) \geqslant 2^{n+1} r^{-j-1}$ for all $1 \leqslant l < k \leqslant N_n$, then

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- If we assume the following concentration inequality

$$\|(\mathbf{E} \sup_{t \in A} Z_t - \sup_{t \in A} Z_t)_+\|_p \leqslant L \sup_{t \in A} \|Z_t\|_p \tag{6}$$

  then one can prove that $F_{n,j}(A) = \mathbf{E} \sup_{t \in A} Z_t$ satisfies (4) and (5).

Canonical processes

- Canonical processes based on e.g. log-concave independent symmetric r.v.'s satisfy Sudakov minoration i.e. if $2d_n(t^l,t^k) \geqslant d_{n+1}(t^l,t^k) \geqslant 2^{n+1}r^{-j-1}$ for all $1 \leqslant l < k \leqslant N_n$, then

$$K\mathbf{E}\sup_{i\leqslant N_n} Z_{t^i} \geqslant 2^{n+1}r^{-j-1}.$$

- If we assume the following concentration inequality

$$\|(\mathbf{E}\sup_{t\in A} Z_t - \sup_{t\in A} Z_t)_+\|_p \leqslant L\sup_{t\in A}\|Z_t\|_p \qquad (6)$$

then one can prove that $F_{n,j}(A) = \mathbf{E}\sup_{t\in A} Z_t$ satisfies (4) and (5).

- Note that we need $(1+\varepsilon)d_n \leqslant d_{n+1} \leqslant 2d_n$

Canonical processes

- Indeed observe that if $A = \bigcup_{i=1}^{N_n} H_i$ and $H_i \subset B_{n+1}(t^i, 2^{n+1} r^{-j-1})$

$$
\begin{aligned}
F(A) &\geqslant K \mathbf{E} \sup_{i \leqslant N_n} (Z_{t^i} + \sup_{t \in H_i} Z_t - Z_{t^i}) \\
&\geqslant K \mathbf{E} \sup_{i \leqslant N_n} Z_{t^i} + \min_{i \leqslant N_n} F(H_i - t^i) \\
&- K(\mathbf{E} \sup_{i \leqslant N_n} (\mathbf{E} \sup_{t \in H_i} Z_t - \sup_{t \in H_i} (Z_t - Z_{t^i})) \\
&\geqslant 2^{n+1} r^{-j-1} + \min_{i \leqslant N_n} F(H_i) \\
&- 2KL \sup_{i \leqslant N_n} \|(\mathbf{E} \sup_{t \in H_i} Z_t - Z_{t^i} - \sup_{t \in H_i} Z_t - Z_{t^i})_+\|_{2^n} \\
&\geqslant 2^{n+1} r^{-j-1} + \min_{i \leqslant N_n} F(H_i) - 2KL \max_{i \leqslant N_n} \Delta_n(H_i).
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- But $\Delta_n(H_i) \leqslant (1+\varepsilon)^{-\kappa} \Delta_{n+\kappa}(H_i) \leqslant \frac{2^{n+2} r^{-j-1}}{(1+\varepsilon)^{\kappa}}$.

Extensions of the result

- Let $Z = (Z_1, Z_2, \ldots, Z_d)$ be isotropic and $Z_t = \sum_{i=1}^{d} t_i Z_i$.

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- **Theorem (WB)**

  Suppose that $\mathbf{P}(|Z_i| > u \,|\, |Z_j|, j \neq i) \leqslant L\mathbf{P}(|Z_i| > u)$ *for all* $u > 0$*, then for any* $T \subset \mathbb{R}^d$ *such that* $|T| \geqslant 2^p$ *and* $\|Z_t - Z_s\|_p \geqslant A$ *we have that*

$$\mathbf{E} \sup_{t \in T} Z_t \geqslant K^{-1} A.$$

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  $$\mathbf{E} \sup_{t \in T} Z_t \geqslant K^{-1} A.$$

- Consequently if $d_n(s,t) = \|Z_t - Z_s\|_{2^n}$, $n \geqslant 1$ satisfy $(1+\varepsilon)d_n \leqslant d_{n+1} \leqslant 2d_n$ then $\mathbf{E}\sup_{t\in T} Z_t$ is comparable with $\gamma_Z(T)$.

Bernoulli comparison

- Recall that $b(T) = \mathbf{E} \sup_{t \in T} X_t$, where $X_t = \sum_{i \geqslant 1} t_i \varepsilon_i$, where ε_i are random signs.

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  - The conjecture holds if $b(T) \sim \gamma_X(T)$.

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- It is known that

$$\|X_t\|_p \sim \sum_{i=1}^{p} |t_i^*| + \sqrt{p}\sqrt{\sum_{i>p} |t_i^*|^2},$$

where $|t_1^*| \geqslant |t_2^*| \geqslant \ldots$ is the non-increasing rearrangement of $|t_1|, |t_2|, \ldots$.

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- **Theorem**
Suppose that for all $s, t \in T$, $p \geqslant 1$ and $r > 0$

$$\|\sum_{i\geqslant 1}(|\varphi(t) - \varphi_i(s)| \wedge r)\varepsilon_i\|_p \leqslant K(rp + \|\sum_{i\geqslant 1}|t_i - s_i| \wedge r)\varepsilon_i\|_p$$

then $b(\varphi(T)) \leqslant Kb(T)$

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- **Corollary**

  Suppose that for any $p \geqslant 1$ and $L \geqslant 1$

$$\inf_{I:|I|\leqslant Lp} \sqrt{\sum_{i \notin I} |\varphi(t)_i - \varphi(s)_i|^2} \leqslant L \inf_{I:|I|=p} (\sum_{i \notin I} |t_i - s_i|^2)^{\frac{1}{2}}$$

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  - For a separable Banach space $(B, \|\cdot\|)$ in order to prove that

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  - does it suffice to show that for any $x^* \in B^*$ and $p \geqslant 1$

$$\|\sum_{i\geqslant 1} x^*(x_i)\varepsilon_i\|_p \leqslant K\|\sum_{i\geqslant 1} x^*(y_i)\varepsilon_i\|_p?$$

Thank you for your attention