Chaining arguments and list decoding

Mary Wootters (based on work with Atri Rudra)

June 23, 2016

1. Error Correcting Codes and List Decoding

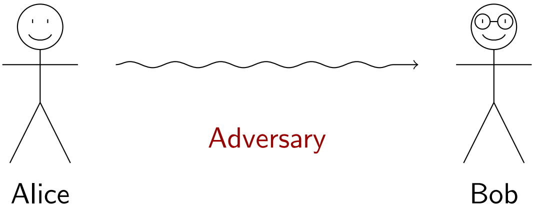
2. Random Codes

3. Setting up a Chaining Argument
   - Average-radius list-decodability
   - Pass to a Gaussian process
   - Controlling the Gaussian process

4. Conclusion

The point of this talk is the chaining argument

Error correcting codes



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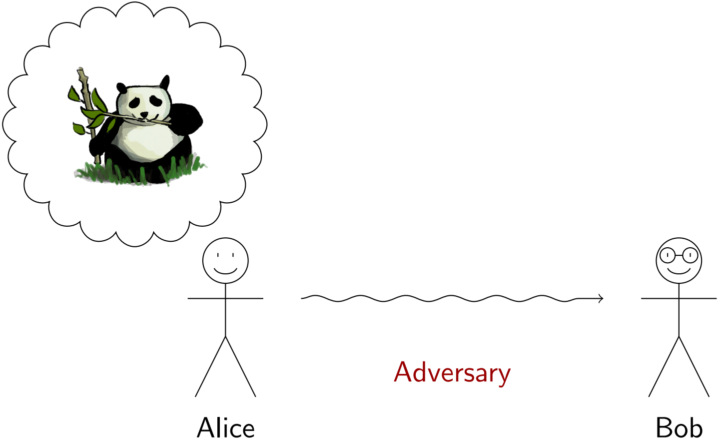

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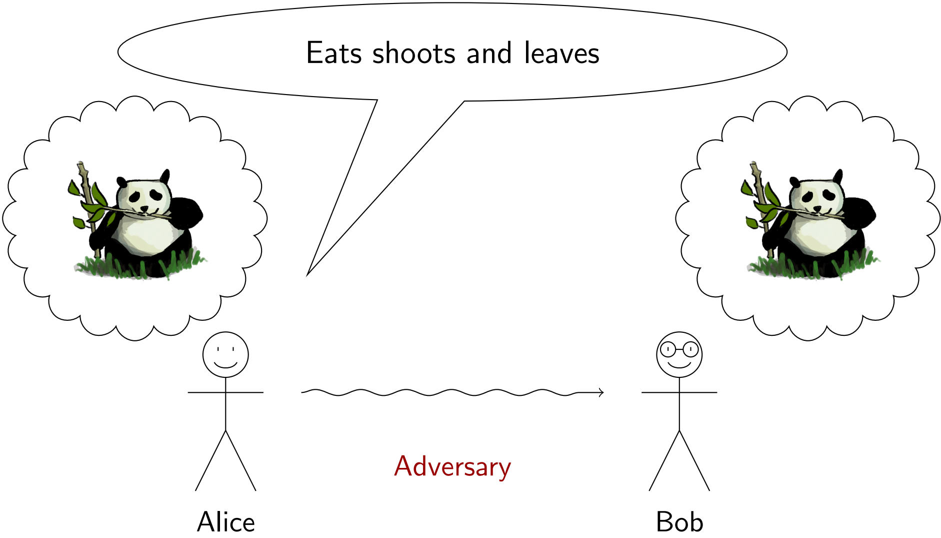

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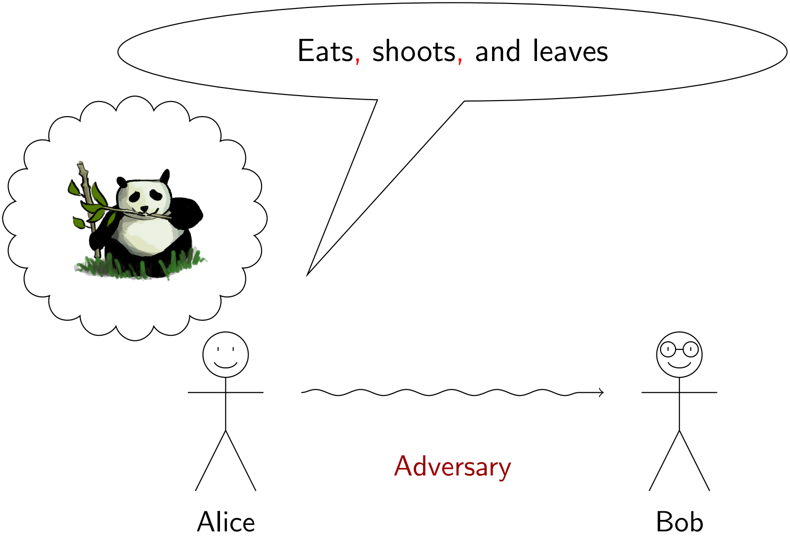

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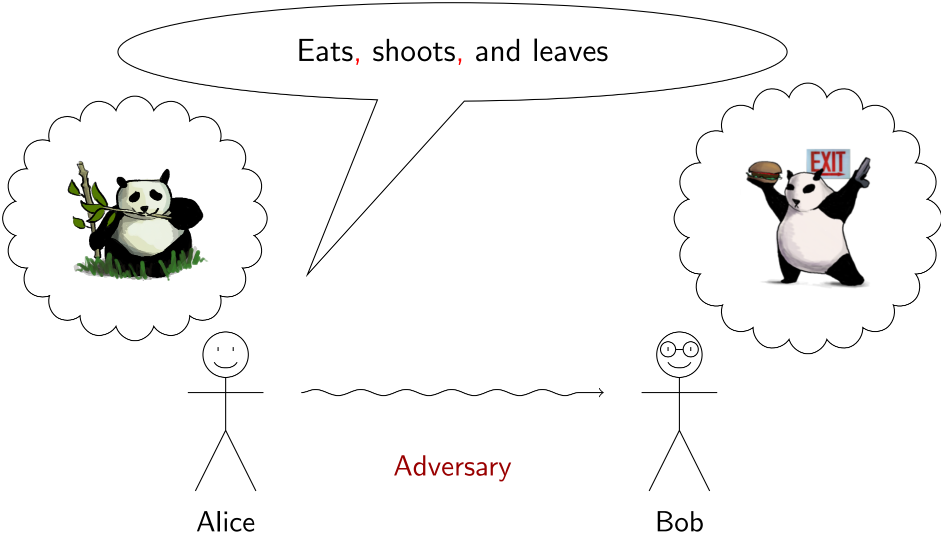

Error correcting codes

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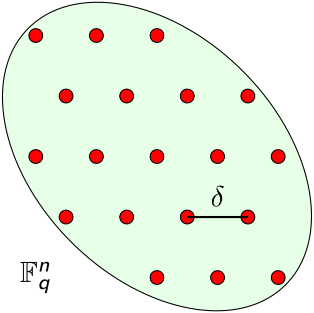

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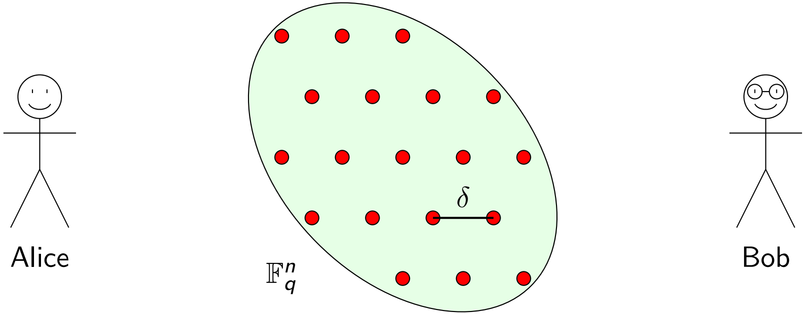

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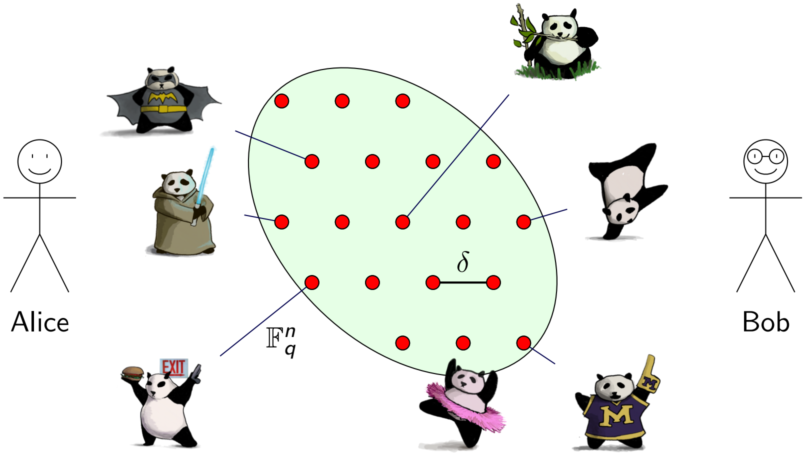

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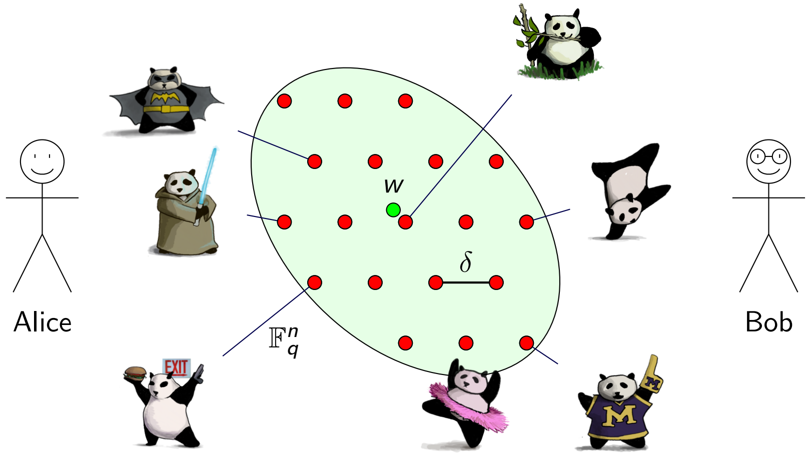

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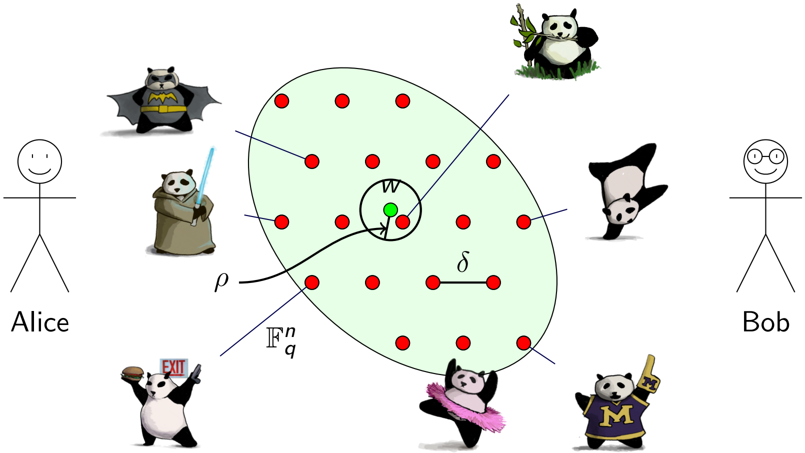
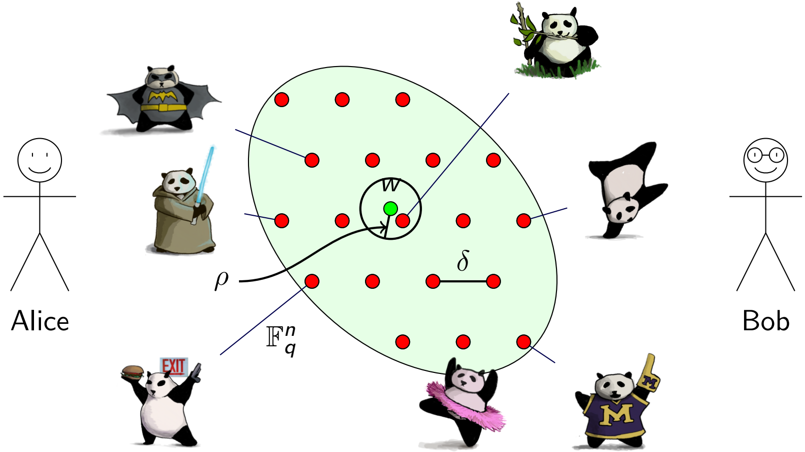


Error correcting codes

- $\mathcal{C} \subset \mathbb{F}_q^n$ is a code, with (relative Hamming) distance δ.
- Alice and Bob can use $\mathcal{C}$ to communicate:
  - error rate is ρ
  - The rate of $\mathcal{C}$ is $R = \log_q |\mathcal{C}|/n$.



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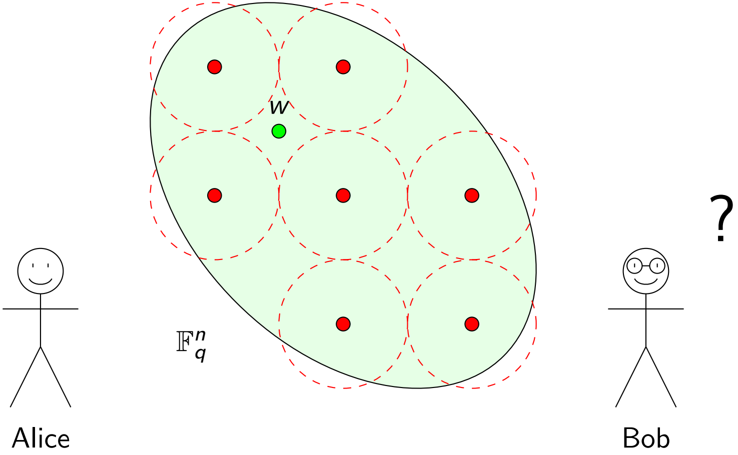
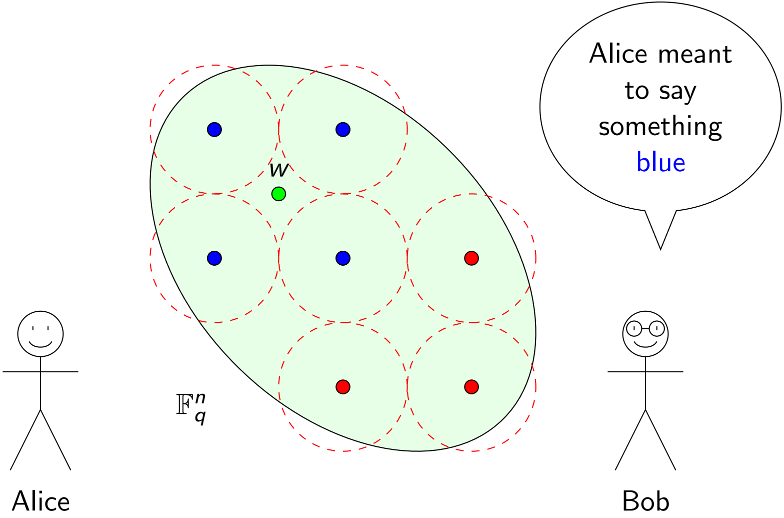
- Bob cannot uniquely decode Alice's message.



But what if the error rate $\rho > \delta/2$?

- Bob cannot uniquely decode Alice's message.
- List decoding to the rescue!

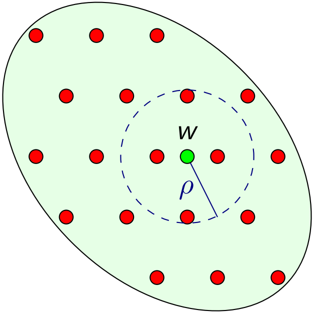


List decodable codes

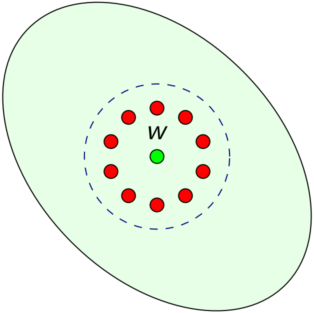
Definition

A code $\mathcal{C} \subset \mathbb{F}_q^n$ is (ρ, L)-list-decodable if

$$\max_{w \in \mathbb{F}_q^n} \left| \{ c \in \mathcal{C} \,:\, d(c, w) \leq \rho \} \right| \leq L.$$



$(\rho, 4)$-list-decodable



not $(\rho, 4)$-list-decodable

Now ρ can be big!

- Turns out (for large q), there are codes which are (ρ, L)-list decododable with:
  - Error rate $\rho = 1 - \varepsilon$
  - Rate $R = \Omega(\varepsilon)$
  - List size $L = O(1/\varepsilon)$

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  - Error rate $\rho = 1 - \varepsilon$
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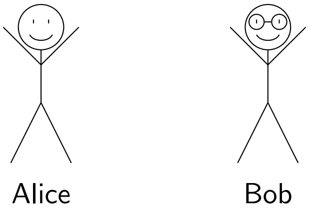
- That means Alice and Bob can win...
  ...even when almost all of the symbols are corrupted!



Not just for Alice and Bob

- Also lots of connections to complexity theory:
  - Hardness amplification
  - Hardcore predicates from one-way functions
  - Extractors
  - Expanders
  - Pseudorandom generators
  - Hardness of computing permanents
- (See [Sudan'00] or [Vadhan'11] for good surveys).

?

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   - Completely random codes
   - Structured random codes
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1. What codes are (near-)optimally list-decodable? (up to $\rho = 1 - \varepsilon$)
   - Completely random codes $\leftarrow$ This talk
   - Structured random codes $\leftarrow$ This talk
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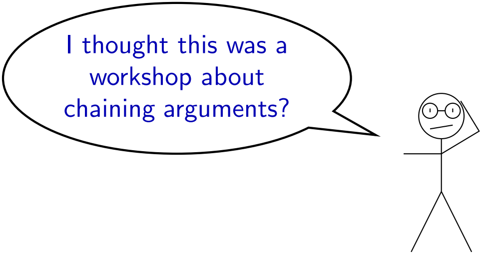
2. Where's the randomness here?



Bob

Structured random codes

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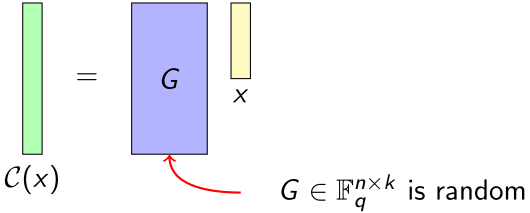
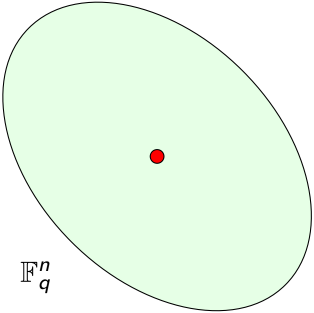


A random linear code $\mathcal{C} \subset \mathbb{F}_q^n$
is a random k-dimensional subspace of $\mathbb{F}_q^n$.

But first! Completely random codes

- Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a completely random code.



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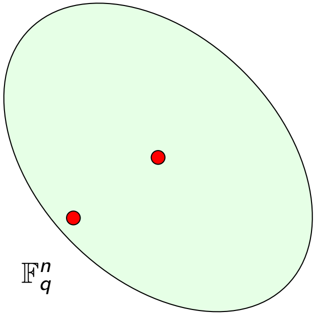

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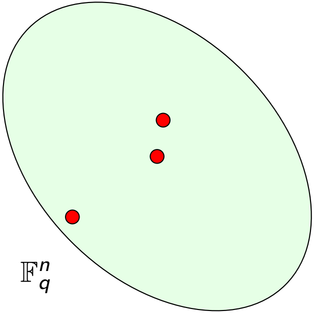

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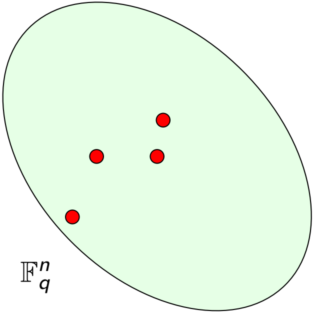

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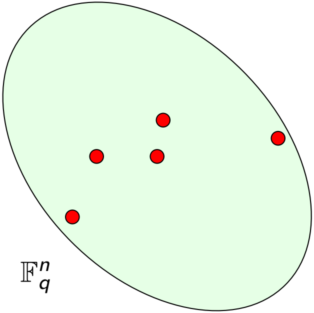

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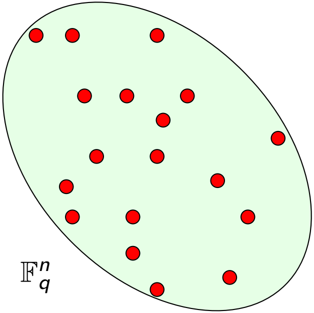

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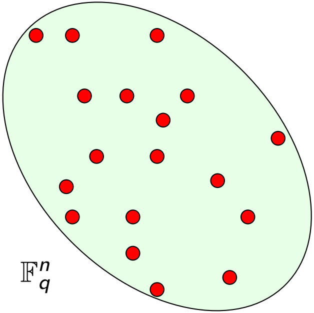

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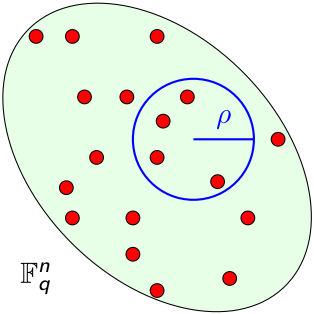

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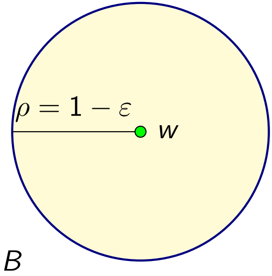

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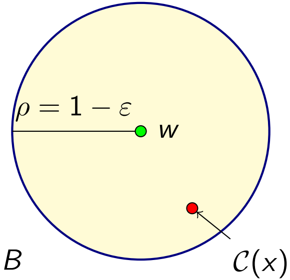

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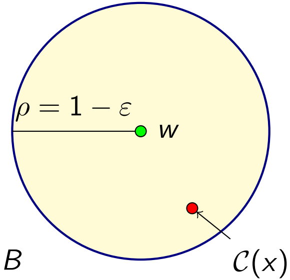

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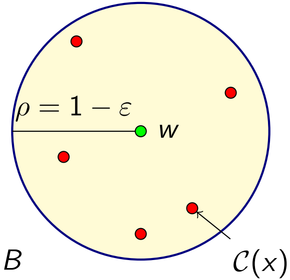

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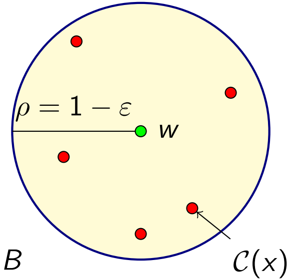

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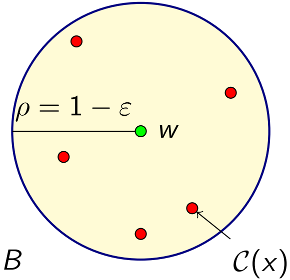

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- Union bound over all w, Λ:

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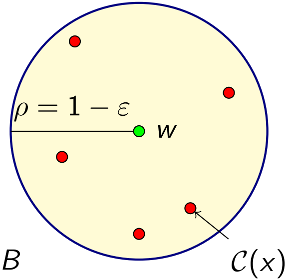

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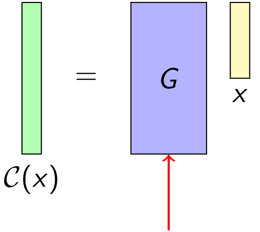
If $L \approx 1/\varepsilon$ and $k/n > \varepsilon$, then this is small!

Try it for random linear codes



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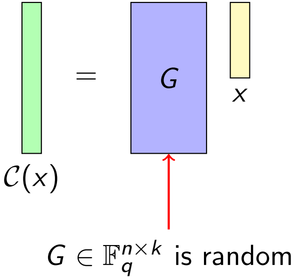

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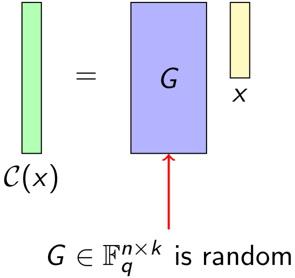

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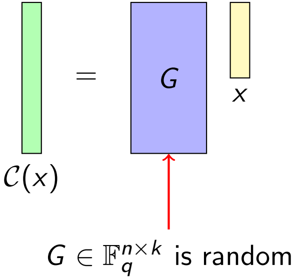

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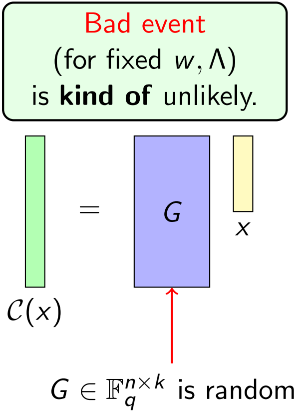

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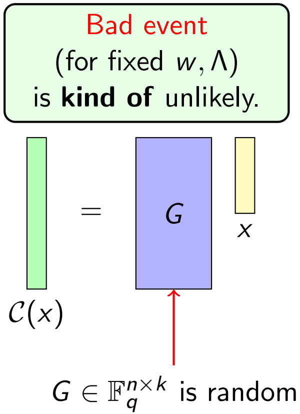

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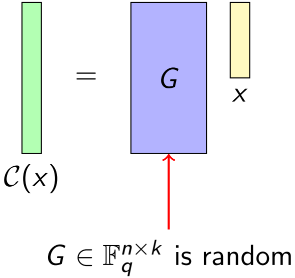

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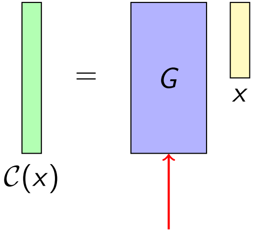

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Question:
Are random linear codes as list-decodable as random codes?

Try it for random linear codes

2-second lit review:

- Asked by [Elias'91].
- small ρ.
  [Guruswami, Hastad, Kopparty'11]
- small q, large ρ.
  [Cheraghchi, Guruswami, Velingker'13],[W.'13]
- large q, large ρ.
  [Rudra-W.'14]

- Fix $w \in \mathbb{F}_q^n$.

$$\mathbb{P}\left\{\mathcal{C}(x) \in B\right\} \leq \text{small} \approx q^{-\varepsilon n}$$

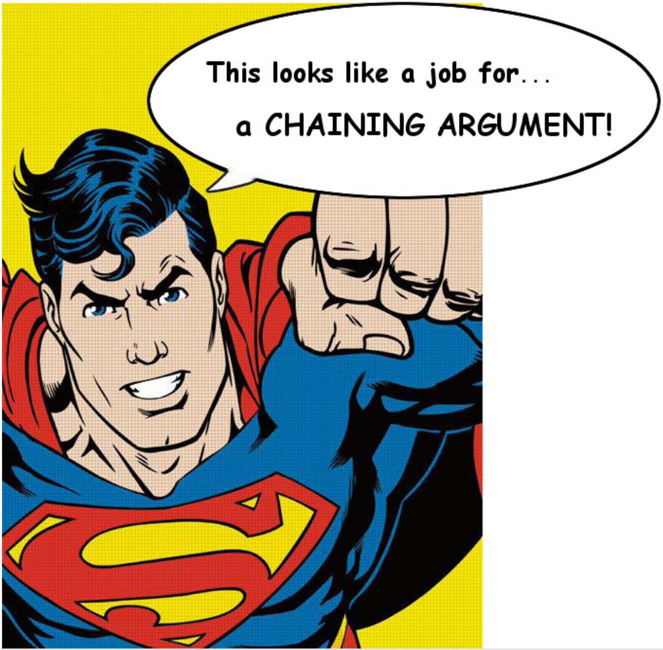
- Fix messages $\Lambda = \{x_0, \ldots, x_L\} \subset \mathbb{F}_q^k$.

$$\mathbb{P}\left\{\mathcal{C}(x_0), \ldots, \mathcal{C}(x_L) \in B\right\} \leq \text{small}^{\log_q(L+1)}$$

- Union bound over all w, Λ...

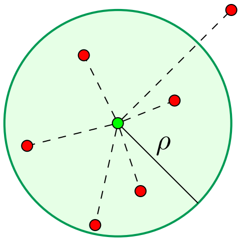
  ...need $L \approx q^{1/\varepsilon}$ to get away with rate $R \approx \varepsilon$

Question:
Are random linear codes as list-decodable as random codes?


This looks like a job for...
a CHAINING ARGUMENT!

Average-Radius List Decodability

List Decodable



All lists of size $> L$ have at least one codeword further than ρ

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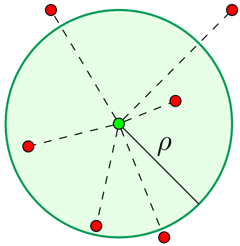

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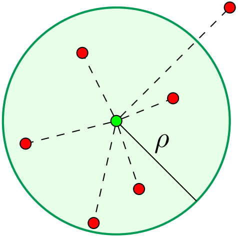
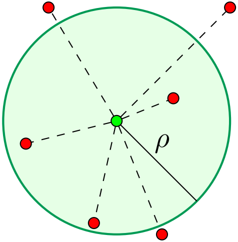


$\Longleftarrow$



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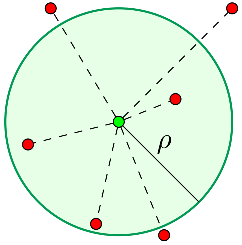

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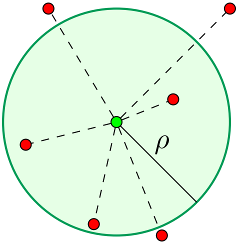
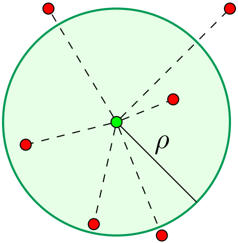


All lists of size $> L$ have average distance larger than ρ.

$$\Leftrightarrow$$

$$\max_{\Lambda \subset \mathbb{F}_q^k, |\Lambda| = L+1} \max_{w \in \mathbb{F}_q^n} \sum_{x \in \Lambda} \mathrm{agr}(w, \mathcal{C}(x)) \leq n \cdot (L+1) \cdot (1-\rho)$$

Average-Radius List Decodability



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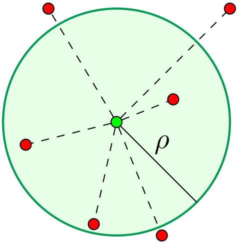

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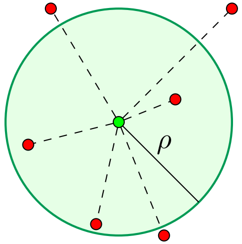

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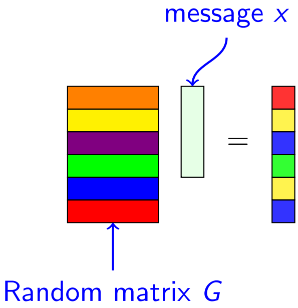

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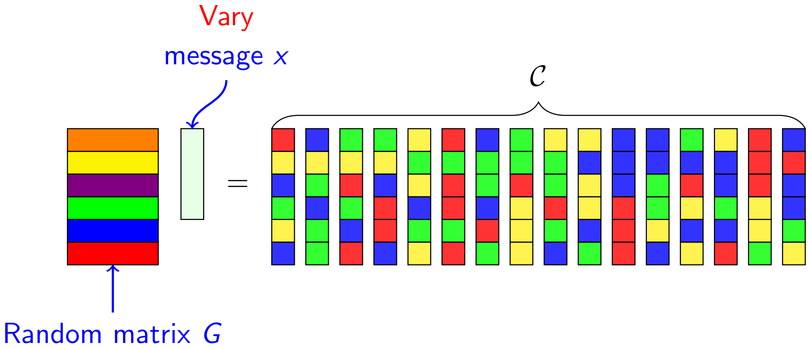

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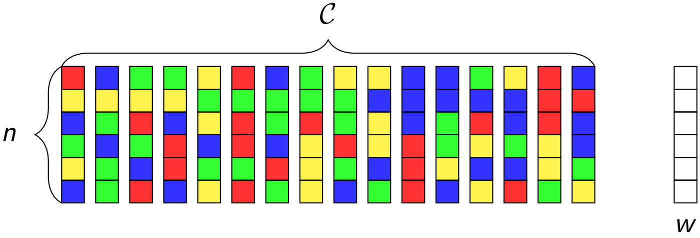

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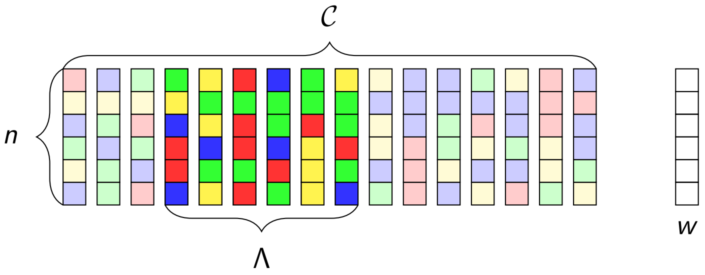

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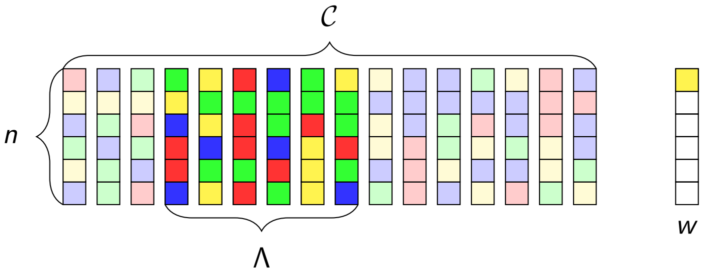

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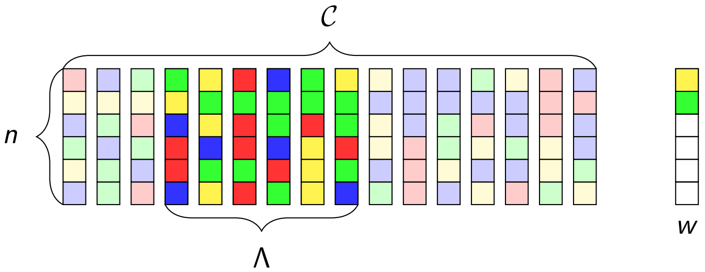

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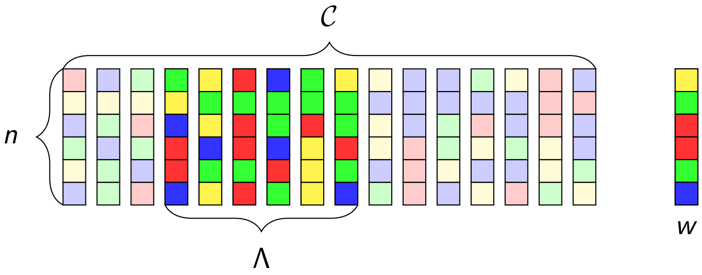

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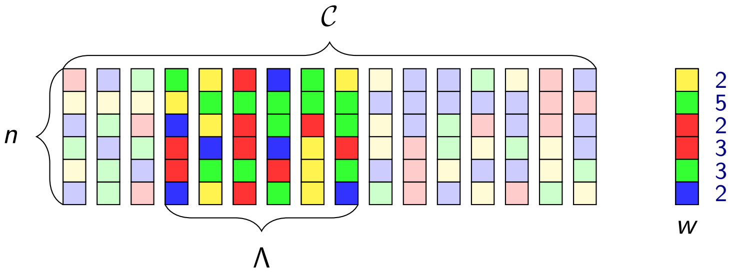

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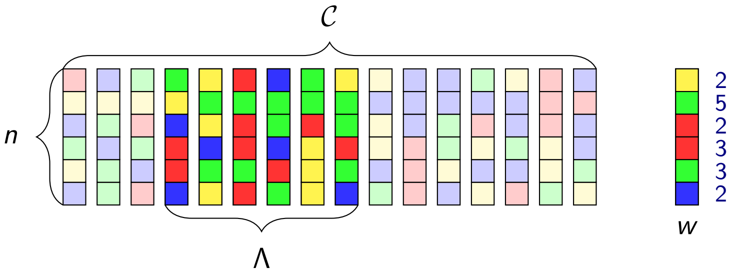

Average agreement of Λ with the worst w:

$$\max_{w \in F_q^n} \sum_{x \in \Lambda} \mathrm{agr}(w, \mathcal{C}(x)) = 2 + 5 + 2 + 3 + 3 + 2 \qquad .$$

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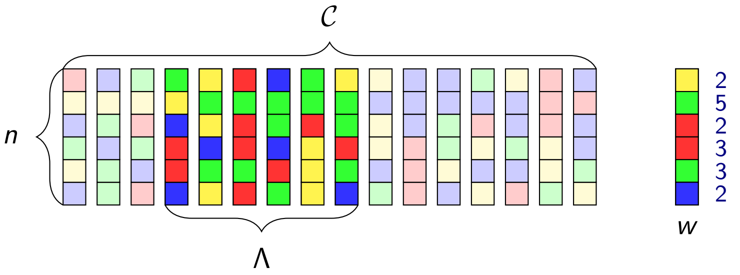

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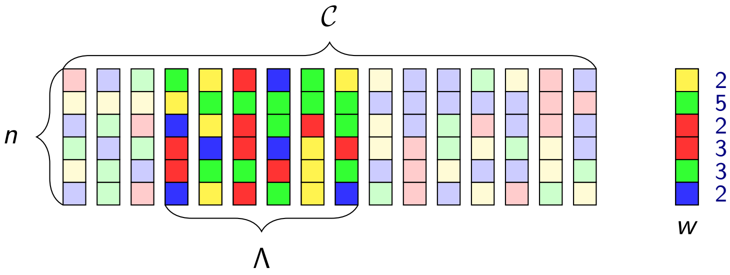

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Independent!

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Yay! A sum of independent random variables

To show that $\mathcal{C}$ is list-decodable, suffices to show that

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Standard tricks

$$\mathbb{E} \max_{|\Lambda|=L} \left| \sum_{j=1}^{n} \mathsf{pl}_j(\Lambda) - \mathbb{E}\mathsf{pl}_j(\Lambda) \right|$$

Standard tricks

Symmetrization and comparison

$$\mathbb{E} \max_{|\Lambda|=L} \left| \sum_{j=1}^{n} \mathrm{pl}_j(\Lambda) - \mathbb{E}\mathrm{pl}_j(\Lambda) \right| \lesssim \mathbb{E} \max_{|\Lambda|=L} \sum_{j=1}^{n} g_j \cdot \mathrm{pl}_j(\Lambda)$$

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Condition on $\mathcal{C}$ until further notice

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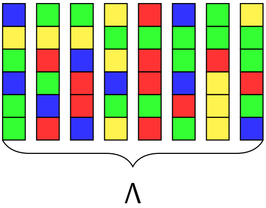

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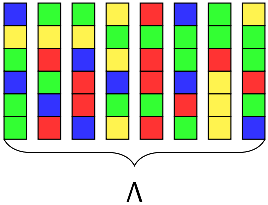

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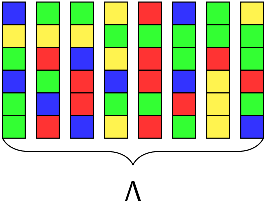

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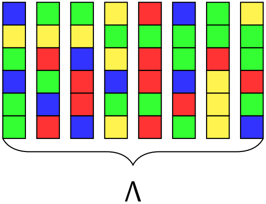

- Natural choice of nets:

$$\mathcal{N}_t = \left\{ \begin{pmatrix} \mathrm{pl}_1(\Lambda) \\ \mathrm{pl}_2(\Lambda) \\ \vdots \\ \mathrm{pl}_n(\Lambda) \end{pmatrix} : |\Lambda| = \frac{L}{2^t} \right\}$$

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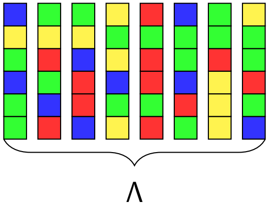

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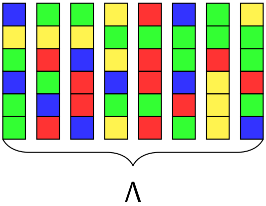

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- Natural way to show that any $\mathrm{pl}(\vec{\Lambda})$ is close to some element of $\mathcal{N}_t$:

  Choose a random subset of Λ of size $L/2^t$.

Why should this work

The goal is to show that $\mathcal{N}_t$ is a decent net of $\mathcal{X}$.
(w.r.t. ℓ_2, where "decent" degrades with t)

$$\mathcal{X} = \left\{ \begin{pmatrix} \mathrm{pl}_1(\Lambda) \\ \mathrm{pl}_2(\Lambda) \\ \vdots \\ \mathrm{pl}_n(\Lambda) \end{pmatrix} : |\Lambda| \leq L \right\} \qquad \mathcal{N}_t = \left\{ \begin{pmatrix} \mathrm{pl}_1(\Lambda) \\ \mathrm{pl}_2(\Lambda) \\ \vdots \\ \mathrm{pl}_n(\Lambda) \end{pmatrix} : |\Lambda| = \frac{L}{2^t} \right\}$$

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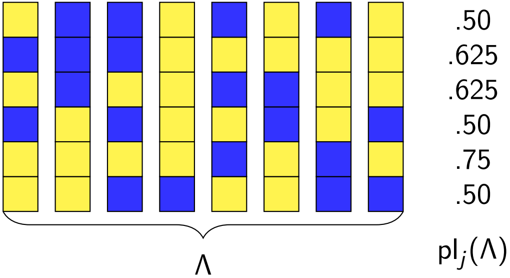

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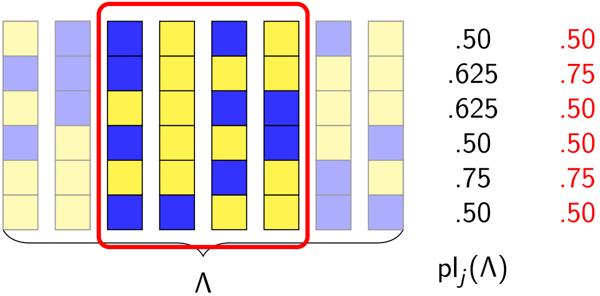

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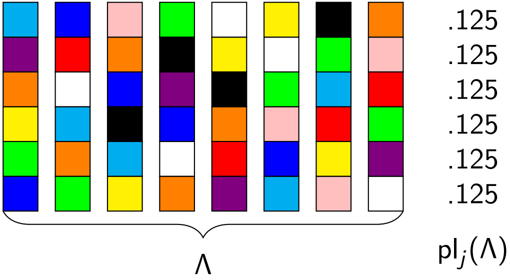

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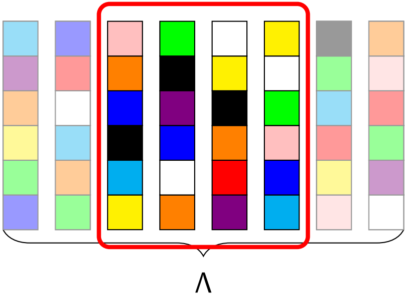

.125	.25
.125	.25
.125	.25
.125	.25
.125	.25
.125	.25

$\mathrm{pl}_j(\Lambda)$

Bummer!

Our great idea

to show that $\mathrm{pl}(\Lambda)$ is small

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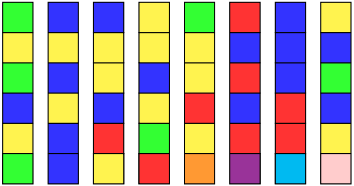
by sets Λ where $\mathrm{pl}(\Lambda)$ is small.

Solution

Nets are made up of pl(Λ), along with a set of active indices



.375
.625
.375
.500
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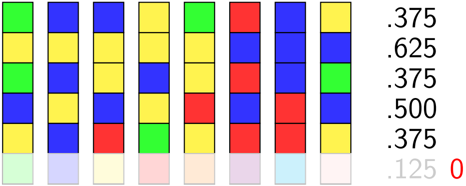
.125

Solution

Nets are made up of pl(Λ), along with a set of active indices



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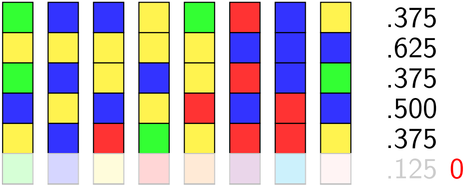

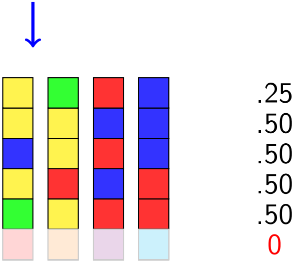

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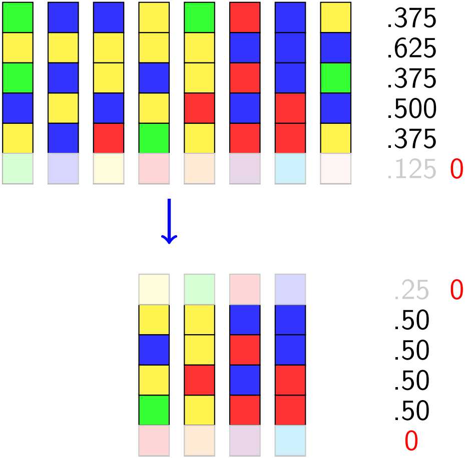

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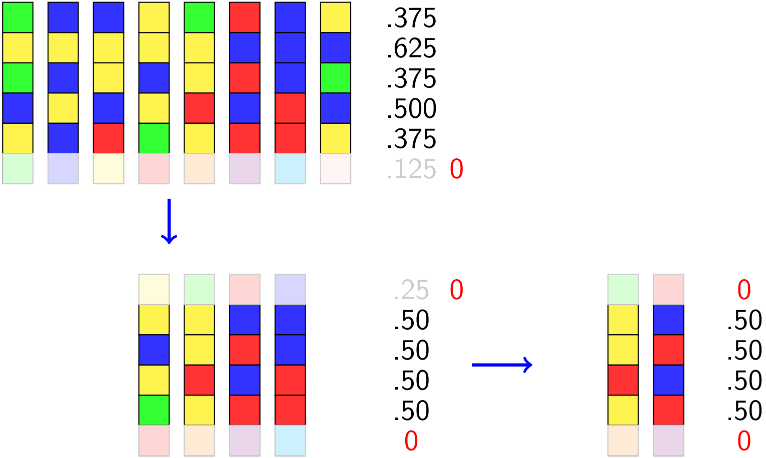

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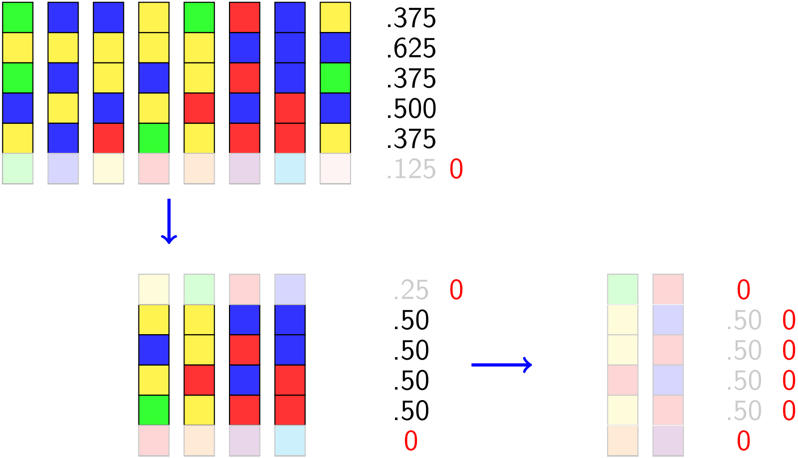
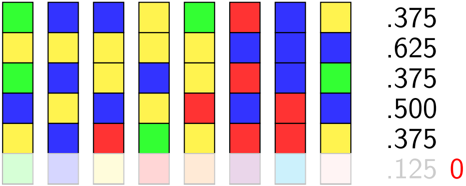


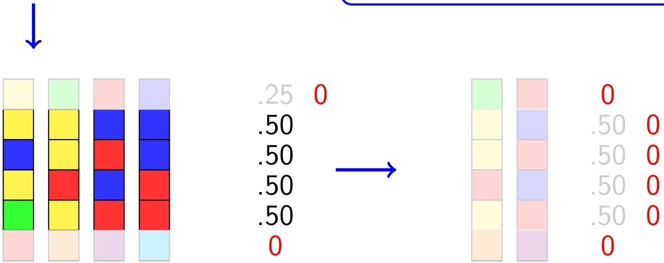
Solution

Nets are made up of $\mathrm{pl}(\Lambda)$, along with a set of active indices



Guarantees:

- The active $\mathrm{pl}_j(\Lambda)$ never get too large.
- The distances between these vectors are not too big.



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$$\|\mathrm{pl}_{I_t}(\Lambda_t) - \mathrm{pl}_{I_{t+1}}(\Lambda_{t+1})\|_2 \lesssim \frac{1}{\sqrt{|\Lambda_t|}} \cdot \sqrt{\sum_{j=1}^{n} \mathrm{pl}_j(\Lambda_0)}.$$

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  - $$\sum_{j \in I_t} \mathrm{pl}_j(\Lambda_t) \leq (1+\eta)^t \left(\sum_{j=1}^{n} \mathrm{pl}_j(\Lambda_0) \right)$$

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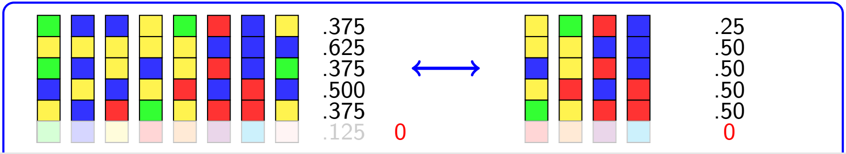

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Call this Q
(this is what we originally wanted to bound)

- Over all of the Λ_0's we could have started with, the number of pairs (Λ_t, I_t) that ever show up at level t is at most $\binom{|\mathcal{C}|}{L/2^t}^2$.

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Chaining argument

$$\boxed{\sum_j g_j \mathsf{pl}_j(\Lambda_0) \approx \sum_{j \in I_T} g_j \mathsf{pl}_j(\Lambda_T)}$$

- $$\|\mathsf{pl}_{I_t}(\Lambda_t) - \mathsf{pl}_{I_{t+1}}(\Lambda_{t+1})\|_2 \lesssim \frac{1}{\sqrt{|\Lambda_t|}} \cdot \sqrt{Q}.$$
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This implies (on board)

$$\mathbb{E} \max_{|\Lambda_0| = L} \left| \sum_{j=1}^n g_j \mathrm{pl}_j(\Lambda_0) - \sum_{j \in I_T} g_j \mathrm{pl}_j(\Lambda_T) \right| \lesssim \sqrt{Q \log |\mathcal{C}|}.$$

Base

$$\boxed{\sum_{j \in I_T} g_j \mathrm{pl}_j(\Lambda_T) \text{ reasonable}}$$

$$\sum_{j \in I_T} \mathrm{pl}_j(\Lambda_T) \lesssim Q \qquad \text{and} \qquad \mathrm{pl}_j(\Lambda_T) \leq 1 \quad \forall j$$

Base

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This implies (in your head)

$$\mathbb{E} \max_{\Lambda_0} \left| \sum_{j \in I_T} g_j \mathrm{pl}_j(\Lambda_T) \right| \lesssim \sqrt{Q}$$

Together

$$\mathbb{E} \max_{|\Lambda_0| = L} \sum_{j=1}^{n} g_j \cdot \mathsf{pl}_j(\Lambda_0) \lesssim \sqrt{Q \log |\mathcal{C}|} = \sqrt{\left(\sum_{j=1}^{n} \mathsf{pl}_j(\Lambda) \right) \log |\mathcal{C}|}$$

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Solving... (and using the fact that $\max \mathbb{E}$ is fine):

$$\mathbb{E} \max_{|\Lambda|=L} \sum_{j=1}^{n} \mathrm{pl}_j(\Lambda) \lesssim n\varepsilon + \log |\mathcal{C}|.$$

Almost done

We just saw

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aka

$$R = \frac{\log_q(\mathcal{C})}{n} \lesssim \frac{\varepsilon}{\log(q)}.$$

Finally

Theorem

Suppose $q \geq \frac{k}{\varepsilon^2}$. Let $\mathcal{C}$ be a random linear code over $\mathbb{F}_q$ with

$$R = \frac{k}{n} = \frac{C\varepsilon}{\log(q)\log^5(1/\varepsilon)}.$$

Then w.h.p, $\mathcal{C}$ is (ρ, L)-list-decodable with

$$\text{error rate } \rho = 1 - \varepsilon \qquad \text{list size } L = O\left(\frac{1}{\varepsilon}\right).$$

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  - Independent symbols.
  - $\max \mathbb{E}$ about right.

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- What did we need from random linear codes?
  - Independent symbols.
  - $\max \mathbb{E}$ about right.
- Also works for Reed-Solomon Codes with random evaluation points.

Recap

- Chaining method for establishing list-decodability of structured random codes.

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Recap

- Chaining method for establishing list-decodability of structured random codes.
- Some punchlines:
  - Random linear codes are (nearly) optimally list-decodable.
  - There are Reed-Solomon codes list-decodable beyond the Johnson bound.
  - More generally, technique works for anything with independent symbols and reasonable $\max \mathbb{E}$.

Many questions

- Remove superfluous log factors?

  the $\log(q)$ is especially obnoxious

- Related: find a simpler proof?

  I mean, I like this one, but...

- Nice characterization of codes list-decodable to capacity?

  Or a nice sufficient condition?

- Applications to other pseudorandom objects?

  Is a random linear extractor optimal?

The end

Thanks for listening!